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MENTAL STRUCTURES CONSTRUCTION ON THE ACE LEARNING

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ABSTRACT: The goal of this study was to analyze the mental structures constructed by students based on the student's initial ability to construct proofs in the ACE learning. The design of this research was descriptive qualitative. The subjects were students who took Introduction to Algebraic Structure 1 course at Universit as Negeri Semarang. The data were collected by test and interview. The credibility of the data was carried out by the technical and source triangulation. Data analysis was data reduction, data display, and conclusion. The results of this study were the mental structure construction of students with the low initial ability for the group was Action, the Schema of group has not been the matized and has not coherent, and the Schema of homomorphism has not been the matized, while the mental structure construction of students with the medium and high initial ability for the group was Object, the Schema of group has been the matized but has not coherent, and the Schema of homomorphism has been the matized.

KEYWORDS: ACE Learning, APOS Theory, Mental Structure.

I. INTRODUCTION

Mathematical proof is a sequence of arguments that justifies a statement is true [1], [2], [3]. Topics in an algebraic structure full of proof that play important roles such as to prove, explain, systematize, discover, communicate, and convince [4]. Individuals who can write valid proofs using mathematical symbols or quantified variables show that the individual has a thorough understanding of the problem [5].

In mathematics education, teaching and learning of deductive proof is one of the most important goals [6]. Teachers should have sufficient knowledge of proof [7]. Gresham (as cited in [8]) argues that the teacher plays an important role in the development of mathematics learning, but many teachers show an inadequate understanding of what supports the proof [9]. The undergraduate Mathematics Education Program at Universitas Negeri Semarang as an institution that educates prospective mathematics teachers provides this provision through one of the mandatory courses, namely Introduction to Algebraic Structure 1, which includes group theory.

Almost everyone includes undergraduate mathematics student experiences difficulty in constructing proof [10], [11], [12], [13]. Because proof forms the basis of mathematical structure, this inability presents problems. Students should be able to understand and construct mathematical proofs [14], [15].

To study group theory, students need mathematical maturity [16] and mastering logic, sets, elements, mappings, equivalence relations, and functions [17], [18]. Referring to this prerequisite, students can be grouped based on their initial ability in constructing proofs on binary operations that contains sets and functions.

Reflective abstraction drives to the construction of the mental structures: Actions, Processes, Objects, and Schemas. Piaget stated (as cited in [19]) that firstly, individuals understood a concept as Action, namely the transformation of the object(s) that were directed externally. When individuals repeated and reflected Actions, they interiorized Actions. They move from relying on external commands to having internal control over the Actions. The interiorized Action is a Process that is characterized by the skill to use symbols, language, and images to understand perceived phenomena [20]. Encapsulation occurs when individuals apply an Action to a Process. Individuals encapsulated the Process into a mental Object. Once a Process has been encapsulated, it can be de-encapsulated to the Process that gives rise to the Objects. Two Processes can be coordinated and the coordinated Process encapsulated into a new Object [19]. According to Dubinsky (as cited in [19]), the Schema is characterized by the dynamics and continuation of the reconstruction that is controlled by the mathematical exercises of the subjects. Thematization is a mental mechanism to obtain a Schema. The coherence of the Schema depends on the understanding of individuals to use it in certain mathematical situations.

The characteristic of the study applying APOS theory is that it begins with a conjecture illustrating the mental structures and mechanisms that students require to learn certain mathematical concepts called genetic

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decomposition. The genetic decomposition of a concept can be different, but describes the actual construction, as expressed in students' work and interview [21], [22].

The ACE learning in this study applies APOS theory, so the first step is to compile genetic decomposition. The refinement of genetic decomposition [23] was used in this ACE learning. There are three stages of the ACE learning, namely Activities, Classroom Discussion, and Exercises.

Some research related to the proofs and application of APOS theory was implemented to examine the ability of prospective teachers to work on formal proofs [24] and statistically tested the ability to validate proofs of students who obtained Abstract Algebra learning based on APOS theory compared to students who obtained conventional learning [25]. Roa-Fuentes and Oktaç (as cited in [19]) conducted interviews to obtain preliminary genetic decomposition which they proposed for the concept of a linear transformation.

The purpose of this study was to analyze the students' mental structures of group in the ACE learning based on students' initial ability to construct proofs.

II. METHOD

This study was a qualitative descriptive involved 36 students who attended Introduction to Algebraic Structure 1 class at Universitas Negeri Semarang. The ACE learning conducted for one semester. The implementation of the ACE learning was as follows.

Activities Stage: doing the STS (Students Task sheet) in a group of 3-5 such that students construct mental structures arranged in the genetic decomposition.

Classroom Discussion Stage: the lecturer was providing definitions and theorems through questions and answers.

Exercises Stage: proving problems in a group of 3-5. The problems help support the development of mental structures construction as stated in genetic decomposition and teach students to practice constructing proofs.

Data collection used tests and interviews. The tests consisted of initial ability test and final test for constructing proofs. The credibility of the data was carried out by the technical and source triangulation. The data were analyzed by data reduction, data display, and conclusion or verification.

III. RESULTS AND DISCUSSION

This study analyzes mental structures constructed for group material through the answers to the problems given. Two problems used were listed below.

Problem 1

Write the definition of a group.

Give an example of a group and prove it.

Problem 2

Write the definition of homomorphism.

Let G be a group and $g \in G$. Show the mapping $\alpha: G \to G$ defined by $\alpha(x) = gxg^{-1}$, $\forall x \in G$ is a homomorphism.

The answer to Problem 1 was intended to analyze whether the Schema of the group has been thematized and was coherent, while the answer to Problem 2 was to explore the constructed mental structure and the coherence of the Schema of the group. A Schema of the group is supported by Schemas of sets, binary operations, and axioms. The following was an analysis of the mental structure constructed by subjects with low (L), medium (M), and high (H) initial abilities.

3.1. Mental Structures Constructed by the Subject with Low Initial Ability (Subject L)

Subject L can write the group definition correctly. He failed in proving the existence of an inverse element. Figure 1 shows the answer to Problem 1 of Subject L related to the existence of an inverse element. The following interview excerpt confirms this finding.

I: You wrote $< R, . >$ as an example of a group. Is that an example of a group?

L: Yes ma'am.

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I: Why?

L: Because it meets all group axioms.

I: Does zero have an inverse?

L: Don't have ma'am. Should be a set of non-zero real numbers, ma'am.

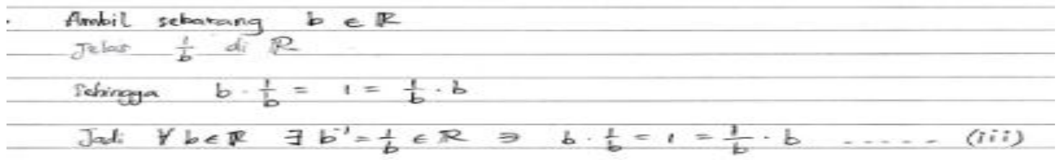

Figure 1. Existence of Inverse Element in Problem 1 of L

Subject L can apply symbols and languages appropriately. He concluded that the set of non-zero real numbers is a group under multiplication after being given an external guide i.e. questions. Based on Figure1 and the interview excerpt above, he did not succeed in verifying the inverse of element property. As a result, he was unsuccessful to coordinate the Processes of the set of real numbers, axioms, and binary operation of multiplication for the existence of inverse elements. In this case, he constructed the mental structure of the Schema for axioms, Action for the set, Process for the binary operation, the Schema of the group has not been thematized and was not coherent.

Subject L did not solve Problem 2. To solve Problem 2 requires the construction of adequate mental structures for the group. The following are excerpts of interviews related to Problem 2.

I: Do you remember the definition of homomorphism?

L: Forgot ma'am...

I: Well, a mapping $\alpha: < G, o > \rightarrow < G,* >$ is called a group homomorphism if $\forall a, b \in G$ satisfy $\alpha(a \ o \ b) = \alpha(a) * \alpha(b)$. Now, show that α in Problem 2 is a homomorphism.

L: (writing) show that α is a homomorphism.

Take any $a, b \in G.$ Then, $\alpha(a) = gag^{-1}$ and $\alpha(b) = gbg^{-1}$.

$\alpha(ab) = gabg^{-1} = abgg^{-1}$.

I: Why?

L: Associative law.

I: Really?

L: Commutative law.

I: Does the commutative law apply?

L: No.

Based on the above interview excerpt, the Subject L did not yet understand the definition of homomorphism and axioms of the group. He cannot use language and symbols in the definition of homomorphism. He has not been able to de-encapsulate the Object of the group into the Process of the axiom as shown in the use of the commutative property for a group so that the coordination of the Processes of mapping, group, and axiom cannot be done properly. Thus, he constructed the mental structure of Action for axioms, Process for the mapping, and Action for the group. He did not thematize the Schema of homomorphism.

From the results of the STS work on the ACE learning Activities stage, the group which the Subject L was in was more likely to make mistakes than did it right. For example, proof of statements that contain universal quantifiers is carried out in certain cases, which is related to proof methods. The interview revealed that he understood the topic presented in the lecture but he was confused when faced with the proving problems. This result is in line with the difficulty associated with the most persistent proof identified among students is the confusion of how to begin the proof [13]. It takes a lot of practice to become familiar with the concept [26], but he did only some problems in the Exercises stage.

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3.2. Mental Structures Constructed by the Subject with Medium Initial Ability (Subject M)

Figure 2 was the answer to Problem 1 of the Subject M related to the existence of an identity element.

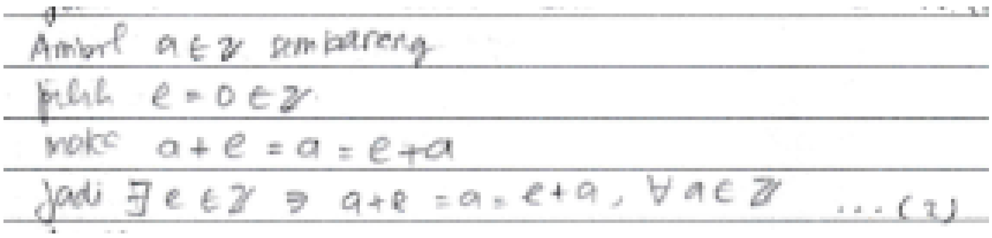

Figure 2. Existence of Identity Element in Problem 1 of M

Subject M can write the definition of a group using symbols correctly and can give an example of a group, namely$< Z,+>$, but she does not provide an explanation of the meaning of the symbol. Figure 2 showed that she failed to prove the axioms of the existence of an identity element. The following interview excerpt reveals the lack of proof methods of Subject M.

I: What is the definition of an identity element?

M: ... it depends on the binary operation. In general operation $*$, $b * e = b = e * b$

I: On the definition of the group, what does the existence of an identity element mean?

M: There is an identity element.

There is $e \in G \ni b * e = b = e * b, \forall b \in G.$

I: How to show it? What are the steps?

M: Take any$b \in G$. Choose $e \in G$ such that later $b * e = b = e * b$.

From the interview excerpt above, Subject M did not understand the meaning of the axiom of the existence of an identity element expressed by mathematical notation concerning operational steps to show it. She can recognize members of the set of integers well and verify the properties of binary operations. In this case, she constructed the mental structure of the Process for axiom, Process for the set, and Object for the binary operation and the Schema of the group has been thematized but it was not coherent.

Figure 3 shows that the Subject M can write the definition of homomorphism using the correct language and symbols and can prove the mapping α is homomorphism but not accompanied by justification at each step. The following interview excerpt shows this finding.

I: Mention the definition of homomorphism.

M: Let $< G,*>$ and $< G', o >$ be groups.

A mapping $\varphi: G \to G'$ is called a group homomorphism if $\forall a, b \in G$ satisfy

$a * b = a \; o \; b$.

I: $a * b = a \; o \; b$?

M: Uhm... $\varphi(a * b) = \varphi(a) \; o \; \varphi(b), \forall a, b \in G$.

I: How to show α is a homomorphism?

M: Take any $x, y \in G$.

Then, $\alpha(x) = gxg^{-1}$ and $\alpha(y) = gyg^{-1}$.

$$\begin{aligned}\alpha(x)\alpha(y) &= gxg^{-1}gyg^{-1}\\ &= gx(g^{-1}g)yg^{-1}\\ &= gxeyg^{-1}\\ &= gxyg^{-1}\end{aligned}$$

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$= \alpha(xy)$

So, $\forall x, y \in G$satisfy $\alpha(xy) = \alpha(x)\alpha(y)$.

Thus, α is a homomorphism.

Figure 3. The Work on Problem 2 of M

Based on Figure 3 and the interview excerpt above, Subject M can state the definition of homomorphism, de-encapsulate Object of a group into the Processes of associative, identity element, and inverse elements, and coordinate the Processes of mapping, group, and axioms. She can prove the mapping α is a homomorphism. In this case, she constructed the mental structure of Schema for the axiom, Process for the mapping, and Object for the group. The Schema of homomorphism has been thematized.

In the Activities stage, Subject M can do the STS well. But in some parts, it shows that she cannot apply the proof method, for example proving statements that contain universal quantifiers by using cases. She believes that it is sufficient to prove an unlimited number of objects by verifying for some specific examples [27]. From the interview, it was obtained the fact that she still had difficulty in carrying out the steps in the proof. This is in line with students experience difficulties with the logic and proof methods [28] and should learn how to construct proofs [29].

3.3. Mental Structures Constructed by the Subject with High Initial Ability (Subject H)

Subject H can write group definition with symbols and mathematical language correctly and can give an example of a group correctly namely $< Z, + >$ but does not describe the symbol. Figure 4 was the answer to Problem 1 of Subject H related to associativity axiom.

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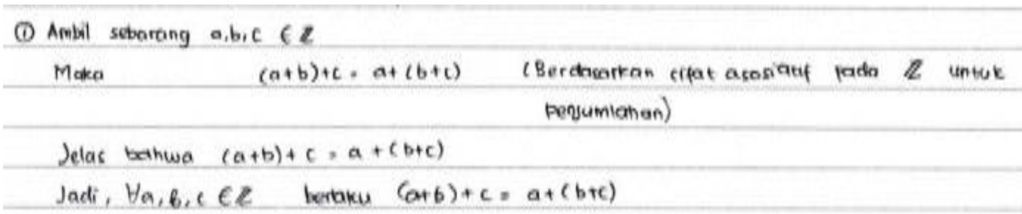

① Ambil sebarang a, b, c ∈ ℤ

Maka $(a+b)+c = a+(b+c)$ (Berdasarkan sifat asosiatif pada ℤ untuk penjumlahan)

Jelas bahwa $(a+b)+c = a+(b+c)$

Jadi, $\forall a,b,c \in \mathbb{Z}$ berlaku $(a+b)+c = a+(b+c)$

Figure 4. Associativity Axiom in Problem 1 of H

In contrast to the existence of the identity element and the existence of the inverse element for addition on the set of integers, she has not succeeded in showing the associative property of integer addition. In showing the associative property for addition on the set of integers, she used the associative property argument itself as shown in Figure 4. The following are excerpts of the interview with the Subject H regarding the associative property of integer addition.

I: What is the definition of a group?

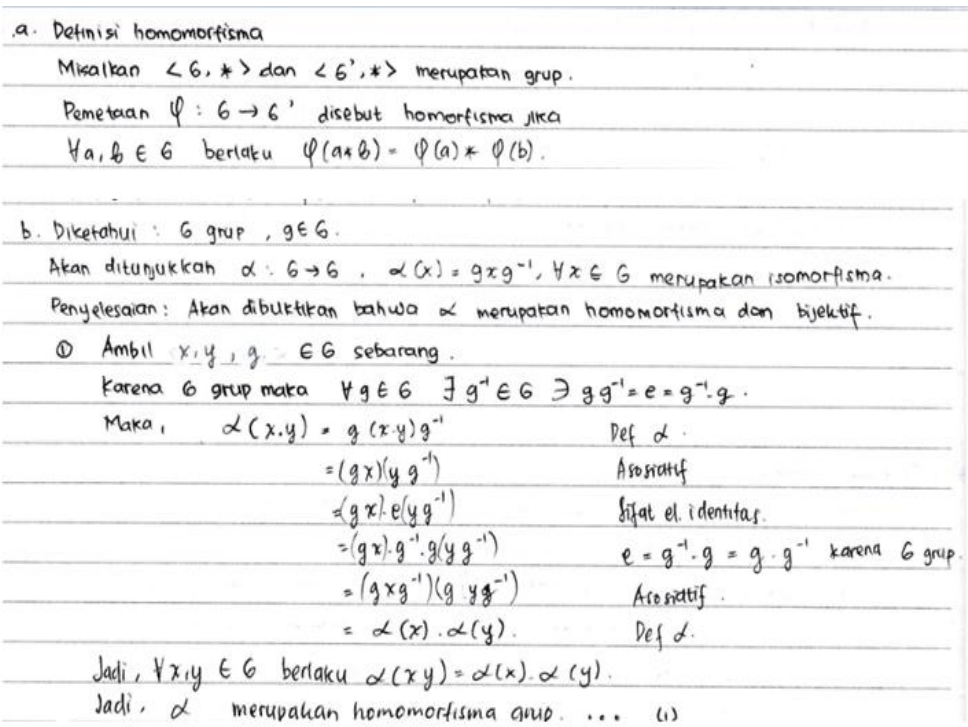
H: Suppose G is a non-empty set and has binary operation$*$. The set G is called a group if the binary operation $*$ is associative on G, there is an identity element e for $*$ on G, and every element in G has an inverse for $*$ on G.

I: Why is the addition associative?

H: Because the addition of the numbers satisfies (a + b) + c = a + (b + c) for each number.

Based on Figure 4 and the interview excerpt above, Subject H can write the definition of a group with the correct language and symbols. She failed to show the associative property. The membership of the integer set can be well recognized. This shows she constructed the mental structure of the Process for axiom, Process for the set of integers, Process for binary operations. The coordination between integer sets, binary operation, and group axioms has been done by the Subject H, but she has not been entirely successful. The Schema of the group has been thematized but has not coherent.

Figure 5 shows the work on Problem 2 of the Subject H.


a. Definisi homomorfisma

Misalkan $\langle G, * \rangle$ dan $\langle G', * \rangle$ merupakan grup.

Pemetaan $\varphi : G \to G'$ disebut homorfisma jika

$\forall a, b \in G$ berlaku $\varphi(a * b) = \varphi(a) * \varphi(b)$.

b. Diketahui: G grup, $g \in G$.

Akan ditunjukkan $\alpha : G \to G$, $\alpha(x) = gxg^{-1}$, $\forall x \in G$ merupakan isomorfisma.

Penyelesaian: Akan dibuktikan bahwa α merupakan homomorfisma dan bijektif.

① Ambil $x, y, g \in G$ sebarang.

Karena G grup maka $\forall g \in G \ \exists g^{-1} \in G \ni gg^{-1} = e = g^{-1}.g$.

Maka, $\alpha(x.y) = g(x.y)g^{-1}$ Def α

$= (gx)(yg^{-1})$ Asosiatif

$= (gx).e(yg^{-1})$ Sifat el. identitas

$= (gx).g^{-1}.g(yg^{-1})$ $e = g^{-1}.g = g.g^{-1}$ karena G grup

$= (gxg^{-1})(gyg^{-1})$ Asosiatif

$= \alpha(x).\alpha(y)$. Def α

Jadi, $\forall x, y \in G$ berlaku $\alpha(xy) = \alpha(x).\alpha(y)$.

Jadi, α merupakan homomorfisma grup ... (i)

Figure 5. The Work on Problem 2 of H

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Subject H can write the definition of homomorphism using the correct language but the binary operation symbol used is incorrect, namely $< G,*>$ and $< G',*>$. She showed the mapping was homomorphism without the quite right method of proof.

Excerpt of interviews about the findings for Subject H was as follows.

I: State the definition of homomorphism.

H: Let $G,*>$ and $< G',*>$ be groups. A mapping $\varphi: G \rightarrow G'$ is called a homomorphism if

$$\varphi(a * b) = \varphi(a) * \varphi(\mathrm{b}), \forall a, b \in G$$

I: Are the group in the domain and the codomain distinct?

H: No ma'am... maybe they are the same.

I: If the group in the domain and the codomain are distinct. Whether the operations are the same?

H: (silent... thinking)...No, ma'am...

I: Not always. Then, the binary operations in groups G and G', do both have to $*$?

H: No ma'am.

I: Ok, let G be a group and $g \in G$.

Defined a mapping $\alpha: G \rightarrow G$ by $\alpha(x) = gxg^{-1}$, $\forall x \in G$. Show that α is a homomorphism.

H: Take any $x, y, g \in G$. Because G is a group then

$\forall g \in G \ \exists g^{-1} \in G \ \ni gg^{-1} = e = g^{-1}g$.

I: In this case, is $g \in G$ fixed?

H: Uhm ... confused.

Based on Figure 5 and the interview excerpt above obtained the following things. Subject H wrote the axioms of homomorphism correctly. Subject H can de-encapsulate the Object of the group into the Processes of associativity, identity element, and inverse elements, apply the mapping rule to any element, as well as coordinate between the Processes of mapping, group, and axiom so that she was successful in proving the mapping α is a homomorphism. Therefore, she constructed the mental structure of Object for the group, the Process for axioms and mapping. She has thematized the Schema of homomorphism.

She can write definitions, but she cannot use language and symbols correctly. This is in line with the finding that students may have difficulty the use of symbols and language as well as proof methods [12] and may not have sufficient understanding of what constitutes proof [30]. In fact, mathematical symbols support readers to immediately grasp ideas in proofs [2].

IV. CONCLUSION

The students' mental structures of the group in the ACE learning based on students' initial ability to construct proofs were as follows.

a. Students with the low initial ability to construct proof constructed the mental structures Action for groups. The Schema of the group has not been yet thematized and was not coherent. The Schema of homomorphism has not been yet thematized. The construction of the mental structures of sets, binary operations, axioms, and mapping were, respectively, Action, Process, Action, and Process.

b. Students with the medium initial ability to construct proof constructed the mental structures of the Object for the group. The Schema of the group has been thematized, but not yet coherent. The Schema of homomorphism has been thematized. The construction of the mental structures of sets, binary operations, axioms, and mapping were respectively, Process, Object, Process, and Process.

c. Students with the high initial ability to construct proof constructed mental structures of the Object for the group. The Schema of the group has been thematized, but not yet coherent. The Schema of homomorphism has been thematized. The construction of the mental structures of sets, binary operations, axioms, and mapping were, respectively, Process, Object, Process, and Process.

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