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Transition probabilities of harmonic oscillator system with spatial Linear-Quadratic-Cubic (LQC) perturbation in time-dependent

Herry F. Lalus [1*] and N P Aryani [2]

[1]Physics Department, Faculty of Teacher Training and Education, Universitas Nusa Cendana Kupang, Indonesia
[2]Physics Department, Universitas Negeri Semarang, Indonesia

*Corresponding author : herrylalus@staf.undana.ac.id

Abstract. We analyze transition probabilities of harmonic oscillator system with spatial LQC (Linear-Quadratic-Cubic) perturbation in time-dependent. This system initially was in the ground state with no perturbation at $t < 0$, then at $t \geq 0$, the system is perturbed by spacial LQC perturbation in time-dependent until $t \to \infty$. We use the time-dependent perturbation theory to analyze this problem. In the initial state, before there is no perturbation, we define the ground state with the base ket of harmonic oscillator without perturbation. Next, when the perturbation is applied to the system, we compute the transition amplitude base on the system state presented above and then we get total wave function that depends on time. By getting this wave function, we can compute transition probability for the system. As a result, there are three transition probabilities, namely the transitions from the ground state to the first, second, and third excited state. There is no transition to others.

1. Introduction

For centuries, scientists all over the world have tried to represent physical phenomena in the language of mathematics. Until now, we may not be sure how many mathematical equations have been generated in the field of physics, such as Classical Mechanics [1], Einstein's Theory of Relativity [2], Quantum Mechanics [3], Quantum Field Theory [4], String Theory [5], and other theories. The form of the resulting equation is very diverse, ranging from the simplest to the most complex. The problem of describing physical phenomena into a mathematical model is certainly a challenge for scientists. On the other hand, the problem of finding solutions to these equations also has its own challenges. There are many problems in physics that we cannot solve precisely because they are related to the complexity of the equations that represent them. In general, only ideal problems can be properly resolved. Therefore, to overcome this problem, it is necessary to use an approximation method that is appropriate and depends on the physical problem being studied [6–10].

In quantum mechanics, things like the ones described above also happen very often. Many problems in quantum mechanics cannot be solved exactly, for example in the case of quantum tunneling effects

for relatively complex potentials [11–14], energy level calculations of many-electron atoms ([15],[16]), Stark effect [17], Zeeman effect [18], hyperfine structure problems [18], emission-absorption of photons by material [3], and so on. To solve these problems, it is very difficult even impossible to obtain exact solutions to the equations. On the other hand, even these systems are sometimes in an impure state as first formulated or studied, but have undergone modification into a more complex form. For example, for a pure multi-electron atomic system, the analysis will be different with the same atom when it is under the influence of an electric field alone, or a magnetic field alone, or a simultaneous magnetic and electric field, or other possibilities, or if the field is constant or changes with respect to time. Not only that, the magnitude of the values of the fields that act on quantum systems will determine what approach/ approximation method will be used to analyze it. The more the physical review, in general, the equations become more difficult and complex, as well as the analysis of the solutions of the equations.
This paper contains our work on the analysis of harmonic oscillator systems perturbed by time-dependent Spatial LQC perturbation. Because perturbation is time-dependent, we use Time-Dependent Perturbation Theory to study it. The subject we examine in this paper is the probability of the transition to quantum states that might occur when the system is perturbed by this type of perturbation. Of course, the solution obtained is not an exact analytic solution but an approximate solution. This paper generally consists of 4 important parts, namely the first part is an introduction, the second part is about Time-Dependent Perturbation Theory which contains Transition Probability, the third part is about the Transition Probability of Perturbed Harmonic Oscillator, and the last part is the conclusion.

2. Time-Dependent Perturbation Theory

2.1. Dyson series

Suppose a quantum system has a state $|\alpha\rangle$ at time $t = t_0$, then at time t, the system is in the state $|\alpha, t_0; t\rangle_I$, where the form $|\quad\rangle_I$ represents the state in the interaction picture. For a quantum system that is subjected to time-dependent perturbation, in the interaction picture, the state equation using the time evolution operator $U_I(t, t_0)$ can be expressed in the form ([3],[19])

$$|\alpha, t_0; t\rangle_I = U_I(t, t_0)|\alpha, t_0; t_0\rangle_I \tag{1}$$

where $U_I(t, t_0) = e^{\frac{iH_0 t}{\hbar}} U(t, t_0) e^{\frac{-iH_0 t_0}{\hbar}}$; $U(t, t_0)$ is time evolution operator in Shcrodinger picture; and H_0 is time-independent Hamiltonian. The time evolution operator in equation (1) satisfies the equation

$$i\hbar \frac{dU_I(t, t_0)}{dt} = V_I(t)U_I(t, t_0) \tag{2}$$

where $V_I(t)$ is a potential interaction picture that has a form

$$V_I(t) = e^{\frac{iH_0 t}{\hbar}} V(t) e^{\frac{-iH_0 t}{\hbar}}. \tag{3}$$

Furthermore, equation (2) becomes

$$\int_{U_I(t,t_0)|t_0}^{U_I(t,t_0)} dU_I(t', t_0) = \frac{1}{i\hbar} \int_{t_0}^{t} V_I U_I(t', t_0)(t')dt'. \tag{4}$$

By using the initial condition $U_I(t, t_0)|_{t=t_0} = 1$, the integral result of equation (4) is obtained, namely

$$U_I(t, t_0) = 1 + \frac{1}{i\hbar} \int_{t_0}^{t} V_I(t')U_I(t', t_0)dt'. \tag{5}$$

Furthermore, assuming that $V_I(t)$ is small, then the solution of equation (5) can be approximated successively. For the first-order approximation, entering $U_I(t', t_0) = 1$, we get

$$U_I^{(1)}(t,t_0) = 1 + \frac{1}{i\hbar}\int_{t_0}^{t} V_I(t')dt'. \tag{6}$$

For the second order approximation, we use $U_I(t',t_0) = U_I^{(1)}(t',t_0)$ and then inserting this form and equation (6) into equation (5) is obtained

$$\begin{aligned} U_I^{(2)}(t,t_0) &= 1 + \frac{1}{i\hbar}\int_{t_0}^{t} V_I(t')\left(1 + \frac{1}{i\hbar}\int_{t_0}^{t'} V_I(t'')dt''\right)dt' \\ &= 1 + \frac{1}{i\hbar}\int_{t_0}^{t} V_I(t')dt' + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^{t} V_I(t')dt' \int_{t_0}^{t'} V_I(t'')dt''. \end{aligned} \tag{7}$$

For the higher orders, we can get it in the same way. Therefore, the general form of this approximation can be expressed in terms of

$$\begin{aligned} U_I(t,t_0) &= 1 + \frac{1}{i\hbar}\int_{t_0}^{t} V_I(t')dt' + \left(\frac{1}{i\hbar}\right)^2 \int_{t_0}^{t} V_I(t')dt' \int_{t_0}^{t'} V_I(t'')dt'' + \cdots \\ &\quad + \left(\frac{1}{i\hbar}\right)^n \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \ldots \times \int_{t_0}^{t^{(n-1)}} dt^{(n)} V_I(t')V_I(t'')\ldots V_I\big(t^{(n)}\big) + \cdots \end{aligned} \tag{8}$$

This form is known as the Dyson Series which is used to calculate the state vector up to the required perturbation order.

2.2. Transition Probability

The transition probability of a quantum system with an initial unperturbed state $|i\rangle$ to state $|n\rangle$ is ([3],[19])

$$P(i \to n) = |\langle n|U_I(t,t_0)|i\rangle|^2 = |C_n|^2 \tag{9}$$

where $n \neq i$, and

$$C_n = C_n^{(0)} + C_n^{(1)} + C_n^{(2)} + \cdots = \langle n|U_I(t,t_0)|i\rangle. \tag{10}$$

By using equation (3), (10) and (8), we find

$$C_n^{(0)} = \delta_{ni} \quad ; \text{(independent of } t\text{)} \tag{11}$$

$$C_n^{(1)} = \frac{1}{i\hbar}\int_{t_0}^{t} \langle n|V_I(t')|i\rangle dt' = \frac{1}{i\hbar}\int_{t_0}^{t} e^{i\omega_{ni}t'}\langle n|V(t')|i\rangle dt' \ ; \tag{12}$$

$$C_n^{(2)} = \left(\frac{1}{i\hbar}\right)^2 \sum_m \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt''\, e^{i\omega_{nm}t'}\langle n|V(t')|m\rangle e^{i\omega_{mi}t''}\langle m|V(t'')|i\rangle. \tag{13}$$

where

$$e^{i\omega_{ni}t} \equiv e^{i(E_n - E_i)t/\hbar}. \tag{14}$$

Equation (9) is a statement of the probability of a quantum state transition up to a certain order in $V_I(t)$. However, for values in high orders generally have a very small contribution to the value of the transition probability in low orders, especially for first-order. Therefore, most of the transition probability analyzes are only carried out in first-order because this order alone is sufficient to represent the physical state of a quantum system such as the problem of atoms and nuclear physics.

3. Transition Probability of Perturbed Oscillator Harmonic System

The quantum system studied in this paper is a one-dimensional harmonic oscillator which has a time-dependent perturbation with the LQC spatial term. The potential form of perturbation is

$$\hat{H}' = (\lambda_1 \hat{x} + \lambda_2 \hat{x}^2 + \lambda_3 \hat{x}^3) e^{-t/\tau} \tag{15}$$

with λ_1, λ_2, and λ_3 are parameters. The total Hamiltonian of the system has the form

$$\hat{H} = \hat{H}_0 + \hat{H}' \text{ ; for } t \geq 0 \tag{16}$$

where

$$\hat{H}_0 = \hbar\omega(\hat{a}^\dagger \hat{a} + \tfrac{1}{2}); \text{ for } t < 0 \tag{17}$$

where $\hat{a}$ and $\hat{a}^\dagger$ are the ladder operators on the harmonic oscillator, respectively.

If we define the initial state ($t < 0$) of this system is $|0\rangle$, then $C_n^{(0)} = \delta_{ni} = \delta_{n0}$, where δ_{ij} is the Delta Kronecker symbol . Furthermore,

$$C_n^{(1)} = \frac{1}{i\hbar} \int_{t_0}^{t} e^{i\omega_{ni} t'} \langle n|V(t')|i\rangle dt' = \frac{1}{i\hbar} \int_{t_0}^{t} e^{i\omega_{ni} t'} V_{ni}(t') dt' \tag{18}$$

where

$$\omega_{ni} = \frac{E_n - E_i}{\hbar}. \tag{19}$$

Then, equation (18) becomes

$$C_n^{(1)} = \frac{1}{i\hbar} \int_{0}^{t} e^{-i(E_0 - E_n)t'/\hbar} \langle n|\hat{H}'(x,t')|0\rangle \, dt' \tag{20}$$

where $V_{n0}(t') \equiv \langle n|\hat{H}'(x,t')|0\rangle$.

Here, what we do is calculating the part $\langle n|\hat{H}'(x,t')|0\rangle$. By using the Hamiltonian of equation (15), then

$$\langle n|\hat{H}'(x,t')|0\rangle = \langle n|(\lambda_1 \hat{x} + \lambda_2 \hat{x}^2 + \lambda_3 \hat{x}^3) e^{-t/\tau}|0\rangle \tag{21}$$

or also expressed in the form

$$\langle n|\hat{H}'(x,t')|0\rangle = (\lambda_1 \langle n|\hat{x}|0\rangle + \lambda_2 \langle n|\hat{x}^2|0\rangle + \lambda_3 \langle n|\hat{x}^3|0\rangle) e^{-t/\tau}. \tag{22}$$

We know that

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^\dagger). \tag{23}$$

Thus,

$$\langle n|\hat{x}|0\rangle = \sqrt{\frac{\hbar}{2m\omega}} \delta_{n1}. \tag{24}$$

For this term, the transition can only occur for $n = 1$. Next, for the quadratic term

$$\langle n|\hat{x}^2|0\rangle = \frac{\hbar}{2m\omega} (\delta_{n0} + \sqrt{2}\delta_{n2}). \tag{25}$$

For this term, without the other terms, the transition is only possible for $n = 0$ and $n = 2$. For the cubic term,

$$\langle n|\hat{x}^3|0\rangle = \left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}} (3\delta_{n1} + \sqrt{6}\delta_{n3}). \tag{26}$$

For this term alone, even without the other terms, the transition only occurs for $n = 1$ and $n = 3$. Since the initial state is $|0\rangle$, the simultaneous perturbation of the three terms allows the transition from the ground state to $n = 1$, $n = 2$, and $n = 3$. From equations (24), (25), and (26), it is obtained

$$C_n^{(1)} = -\frac{i}{\hbar}\int_0^t e^{-\frac{i(E_0-E_n)t'}{\hbar}} \left(\frac{\hbar}{2m\omega}\lambda_2\delta_{n0} + \sqrt{\frac{\hbar}{2m\omega}}\delta_{n1}\left(\lambda_1 + \frac{3\hbar}{2m\omega}\lambda_3\right) + \sqrt{2}\frac{\hbar}{2m\omega}\lambda_2\delta_{n2} + \sqrt{6}\left(\frac{\hbar}{2m\omega}\right)^{3/2}\lambda_3\delta_{n3}\right) e^{-t'/\tau}dt' \tag{27}$$

As explained earlier that in the initial state, $C_n^{(0)} = \delta_{ni} = \delta_{n0}$ is obtained, so for this condition we get

$$C_0^{(0)} = 1; C_1^{(0)} = 0 = C_2^{(0)} = C_3^{(0)}. \tag{28}$$

For the state after being subjected to perturbation, we first calculate for $C_0^{(1)}$, that is

$$C_0^{(1)} = -\frac{i}{\hbar}\int_0^t \frac{\hbar}{2m\omega}\lambda_2\, e^{-\frac{t'}{\tau}}dt' = \frac{i\lambda_2\tau}{2m\omega}\left(e^{-\frac{t}{\tau}} - 1\right) \tag{29}$$

For $t \gg \tau$,

$$C_0^{(1)} = -\frac{i\lambda_2\tau}{2m\omega}. \tag{30}$$

Next, we calculate $C_1^{(1)}$, that is

$$\begin{aligned} C_1^{(1)} &= -\frac{i}{\hbar}\sqrt{\frac{\hbar}{2m\omega}}\int_0^t e^{-\frac{i(E_0-E_1)t'}{\hbar}}\left(\lambda_1 + \frac{3\hbar}{2m\omega}\lambda_3\right) e^{-\frac{t'}{\tau}}dt' \\ &= -\frac{i}{\sqrt{2m\omega\hbar}}\left(\lambda_1 + \frac{3\hbar}{2m\omega}\lambda_3\right)\frac{1}{i\omega - \frac{1}{\tau}}\left(e^{\left(i\omega-\frac{1}{\tau}\right)t} - 1\right). \end{aligned} \tag{31}$$

For $t \gg \tau$,

$$C_1^{(1)} = \frac{i}{\sqrt{2m\omega\hbar}}\left(\lambda_1 + \frac{3\hbar}{2m\omega}\lambda_3\right)\frac{1}{i\omega - \frac{1}{\tau}}. \tag{32}$$

Then, we calculate $C_2^{(1)}$, and by using the same way, we find

$$C_2^{(1)} = -\frac{i\sqrt{2}\lambda_2}{2m\omega}\frac{1}{2i\omega - \frac{1}{\tau}}\left(e^{\left(2i\omega-\frac{1}{\tau}\right)t} - 1\right) \tag{33}$$

For $t \gg \tau$,

$$C_2^{(1)} = \frac{i\sqrt{2}\lambda_2}{2m\omega}\frac{1}{2i\omega - \frac{1}{\tau}}. \tag{34}$$

Furthermore, we calculate $C_3^{(1)}$, and by using the same way, we find

$$C_3^{(1)} = \frac{i\sqrt{6}\lambda_3}{\hbar}\left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\frac{1}{\frac{1}{\tau} - 3i\omega}\left(e^{\left(3i\omega-\frac{1}{\tau}\right)t} - 1\right). \tag{35}$$

For $t \gg \tau$,

$$C_3^{(1)} = -\frac{i\sqrt{6}\lambda_3}{\hbar}\left(\frac{\hbar}{2m\omega}\right)^{\frac{3}{2}}\frac{1}{\frac{1}{\tau}-3i\omega}. \tag{36}$$

State ket for $t \gg \tau$ is

$$|\alpha, t_0 = 0; t\rangle = \sum_n C_n(t)e^{-iE_n t/\hbar}|n\rangle. \tag{37}$$

Thus, the total state for this system is

$$|\alpha, t_0 = 0; t\rangle = C_0(t)e^{-\frac{iE_0 t}{\hbar}}|0\rangle + C_1(t)e^{-\frac{iE_1 t}{\hbar}}|1\rangle + C_2(t)e^{-\frac{iE_2 t}{\hbar}}|2\rangle + C_3(t)e^{-\frac{iE_3 t}{\hbar}}|3\rangle \tag{38}$$

where E_n is the energy for the oscillator harmonic system, $E_n = \hbar\omega\left(n+\frac{1}{2}\right)$; and $C_n(t) = C_n^{(0)} + C_n^{(1)}$. By utilizing these two forms, equation (38) can be expressed in terms of

$$\begin{aligned}|\alpha, t_0 = 0; t\rangle = &\left(1-\frac{i\lambda_2\tau}{2m\omega}\right)e^{-\frac{i\omega t}{2}}|0\rangle - \frac{i}{\sqrt{2m\omega\hbar}}\left(\lambda_1 + \frac{3\hbar}{2m\omega}\lambda_3\right)\frac{1}{\frac{1}{\tau}-i\omega}e^{-\frac{3i\omega t}{2}}|1\rangle \\ &- \frac{i\sqrt{2}\lambda_2}{2m\omega}\frac{1}{\frac{1}{\tau}-2i\omega}e^{-\frac{5i\omega t}{2}}|2\rangle - \frac{i\sqrt{6}\lambda_3}{\hbar}\left(\frac{\hbar}{2m\omega}\right)^{3/2}\frac{1}{\frac{1}{\tau}-3i\omega}e^{-\frac{7i\omega t}{2}}|3\rangle\end{aligned} \tag{39}$$

From this wave function, it can be seen that the system has the possibilities for transitions to the state $|1\rangle$, $|2\rangle$,, and $|3\rangle$, that is

$$P_1 = \frac{\frac{\tau^2(3\hbar\lambda_3 + 2\lambda_1 m\omega)^2}{8m^3\omega^3(\hbar + \hbar\tau^2\omega^2)}}{1 + \frac{\lambda_2^2\tau^2}{4m^2\omega^2} + \frac{\tau^2(3\hbar\lambda_3 + 2\lambda_1 m\omega)^2}{8m^3\omega^3(\hbar + \hbar\tau^2\omega^2)} + \frac{\tau^2\lambda_2^2}{2m^2\omega^2 + 8m^2\tau^2\omega^4} + \frac{3\hbar\lambda_3^2\tau^2}{4m^3\omega^3(1+9\tau^2\omega^2)}}; \tag{40}$$

$$P_2 = \frac{\frac{\tau^2\lambda_2^2}{2m^2\omega^2 + 8m^2\tau^2\omega^4}}{1 + \frac{\lambda_2^2\tau^2}{4m^2\omega^2} + \frac{\tau^2(3\hbar\lambda_3 + 2\lambda_1 m\omega)^2}{8m^3\omega^3(\hbar + \hbar\tau^2\omega^2)} + \frac{\tau^2\lambda_2^2}{2m^2\omega^2 + 8m^2\tau^2\omega^4} + \frac{3\hbar\lambda_3^2\tau^2}{4m^3\omega^3(1+9\tau^2\omega^2)}}; \tag{41}$$

$$P_3 = \frac{\frac{3\hbar\lambda_3^2\tau^2}{4m^3\omega^3(1+9\tau^2\omega^2)}}{1 + \frac{\lambda_2^2\tau^2}{4m^2\omega^2} + \frac{\tau^2(3\hbar\lambda_3 + 2\lambda_1 m\omega)^2}{8m^3\omega^3(\hbar + \hbar\tau^2\omega^2)} + \frac{\tau^2\lambda_2^2}{2m^2\omega^2 + 8m^2\tau^2\omega^4} + \frac{3\hbar\lambda_3^2\tau^2}{4m^3\omega^3(1+9\tau^2\omega^2)}}. \tag{42}$$

There is no probability for transition to other states such as $|4\rangle$, $|5\rangle$, ..., and so on. We can see in that the spatial quadratic perturbation in the harmonic oscillator system produces a transition probability to state $|2\rangle$, and there is no transition probability to another state, whereas when we add the linear and cubic perturbation terms to the quadratic perturbation, the total perturbation becomes LQC, then it can be seen from equations (40), (41), and (42) that the transition of quantum states becomes possible to states $|1\rangle$, $|2\rangle$,, and $|3\rangle$. This clearly looks interesting because the perturbation order is in fact largely determines the result of the transition probability to the corresponding state. It can be seen that a spatial first-order perturbation allows a transition to a $|1\rangle$ state, a second-order perturbation allows a transition to a $|2\rangle$ state, and a third-order perturbation allows a transition to a $|3\rangle$ state. Thus, for a j-order spatial perturbation to a pure harmonic oscillator system in the ground state will produce a transition probability to state $|j\rangle$. It is clear that this is a very interesting conclusion.

From equations (40), (41), and (42) it can be seen that for the first-order, second-order, and third-order perturbations imposed on the oscillator system, the transition probability value P_1 contains λ_1, λ_2, and λ_3; likewise with P_2 and P_3. This shows that the transition probability to each state is also influenced by the perturbation of other orders. However, from the form of these equations, we can see that when the perturbation of other orders is omitted, the transition probability for the particular order under consideration will still maintain the probability of the transition to the state corresponding to the perturbation order. Suppose we look at P_1, when the second-order and the third-order perturbations are omitted, P_1 has a value that only depends on the first-order perturbation. This can be seen from the parameter λ_1 which remains at P_1 when λ_2 and λ_3 are chosen to be zero. Likewise, for P_2 and P_3, we can apply the same thing.

4. Conclusion

We have performed a transition probability analysis of a harmonic oscillator quantum system subjected to time-dependent perturbation with spatial terms of first-order, second-order, and third-order simultaneously. We get the result that there are some probabilities of transition from the ground state to the some states, which have their explicit forms. We also find that, it turns out that the probability of the transition to a certain state depends on the order of the perturbation, that is, if there is a perturbation of the j-order, there will be a probability of a transition to state $|j\rangle$, whether the perturbation is a single perturbation of a certain spatial function or the simultaneous perturbation of several spatial functions of the different orders.

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