The total irregularity strength of caterpillars with odd number of internal vertices of degree three

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The total irregularity strength of caterpillars with odd number of internal vertices of degree three

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Abstract. Given a graph G consisting of vertex set V and edget set E, repectively. Assume G is simple, connected, and the edges do not have direction. A function λ that maps $V \cup E$ into a set of k-integers is named a totally irregular total k-labelling if no vertices have the same weight and also the edges of G get distinct weights. We call the minimum number k for which G has totally irregular total k-labelling as total irregularity strength of G, ts(G). In this article, we construct labels of vertices and edges of caterpillar graphs which have q internal vertices of degree 3 where q is 5,7, and 9. We obtain the exact values of ts in the following: $n + 2$ if the caterpillars have q=5 internal vertices, $n + 3$ for q=7, and $n + 4$ for q=9.

1. Introduction

The notion of graph labelling was introduced in Alexander Rosa's paper in 1967. The definition is as follows: "graph labelling is a function that has elements of $G(V, E)$ as its domain and a set of positive integers as its co-domain. If the function assigns $V(G)$ to the co-domain, then it is called a *vertex labelling*. Meanwhile, if the function maps $E(G)$ to the set of positive integers, then it is named an *edge labelling*. Moreover, the function is mentioned as a *total labelling* when it has $V \cup E$ as its domain" [1].

Further, the definition of *edge irregular total labelling* was given in Baca et al. as follows: "a total k-labelling $f : V \cup E \to \{1, 2, \cdots, k\}$ is named as an *edge irregular total k-labelling* when $w(e_1) \neq w(e_2)$ for each different pair of edges e_1, e_2 in $E(G)$. If $e = xy$, then the weight $w(e) = f(x) + f(y) + f(xy)$. The minimum number k in such a way G has such labelling as a total edge irregularity strength of G, indicated by *tes*(G)" [2]. Whereas, the tes of a tree $T(V, E)$ with a maximum degree $\Delta(T)$ was proved in [3]:

$$tes(T) = max\left\{\left\lceil\frac{|E(T)| + 2}{3}\right\rceil, \left\lceil\frac{\Delta(T) + 1}{2}\right\rceil\right\} \quad (1)$$

Note that the symbol $\lceil x \rceil$ means the least integer greater than or equal to x.

The definition of *vertex irregular total k-labelling* was also given in Baca et al. as follows: "a total k-labelling $f : V \cup E \to \{1, 2, \cdots, k\}$ is named as an *vertex irregular total k-labelling* if the weights $w(v_1) \neq w(v_2)$ whenever $v_1 \neq v_2$ in $V(G)$, where $w(v) = f(v) + \sum_{vy \in E} f(vy)$. We call the minimum positive integer k for which G has such labelling as a total vertex irregularity strength of G, symbolized by $tvs(G)$" [1]. The tvs of a *tree* $T(V,E)$ that consists of n pendant vertices and it does not have vertices of degree 2 was proved in [4]:

$$tvs(T) = \left\lceil \frac{n+1}{2} \right\rceil \qquad (2)$$

Furthermore, the notion of *totally irregular total labelling* was initiated in [5]: "the function f is called out as a totally irregular total k-labelling if both of the vertex-weights and the edge-weights are distinct. The minimum integer k such that G has totally irregular total k-labelling is named as total irregularity strength of G, denoted by $ts(G)$". The lower bound for ts of any graph was also given in [5]:

$$ts(G) \geq max\{tes(G), tvs(G)\} \qquad (3)$$

Some researchers have found tes and tvs of any graph classes as in [6], [7], [8-13], and [14]. Whereas, several exact values of ts of any graph have also been found, such as in [14-18]. A *caterpillar* graph is a *tree* such that removing its pendant edges will form a path [19]. The exact values of ts of some classes of caterpillars are still unknown [15]. Therefore, we investigate ts of caterpillar graphs where the number of internal vertices of degree 3 is odd.

2. Methods

According to the method in [2],[15-18], we summarize the steps used in this paper as follows:

a. Defining caterpillars which have q internal vertices of degree 3 where $q = 5,7,9$. The caterpillar is denoted by $S_{n,\underbrace{3,3,\dots,3}_{q},n}$.
b. Calculating tes of the caterpillars based on eq.(1).
c. Calculating tvs of the caterpillars based on eq.(2).
d. Determining lower bound of ts of $G = S_{n,\underbrace{3,3,\dots,3}_{q},n}$ according to inequality (3).
e. Setting the lower bound $k = \max\{\text{tes}(G), \text{tvs}(G)\}$ where $\text{tes}(G)$ is given in (1) and $\text{tvs}(G)$ is provided in (2), respectively.
f. Proving that ts of the caterpillars is least than or equal to k by constructing a totally irregular total k- labelling f (trial and error process) on the caterpillars and the process is continued until we obtain a fixed pattern of the labelling;
g. Formulating vertex and edge-labels and formulating the ts of the caterpillars.
h. Proving the formula of the weights and showing that the weights are different.
i. Obtaining the $ts = k$.

3. Results and Discussion

We present the results of ts of caterpillar graphs having odd number of internal vertices of degree 3.

3.1. Caterpillar graphs with 5 internal vertices of degree 3

We present the concept of the caterpillars in Definition 3.1.1.

Definition 3.1.1. C*aterpillar* $S_{n,\underbrace{3,3,\dots,3}_{5},n}$ is tree which is formed from *double star* $S_{n,n}$ by inserting five vertices $(v^2, v^3, v^4, v^5, v^6)$ on the bridge that connects the centrals of the stars (v^1 and v^7) and the five vertices are incident to pendant edges $(v_1^j | 2 \leq j \leq 6\}$. Whereas, the vertices of the double stars are $\{v_i^1 : 1 \leq i \leq n-1\}$ and $\{v_i^7 : 1 \leq i \leq n-1\}$. The caterpillar has $2n+10$ vertices, $2n+9$ edges, and $2n+3$ pendant vertices. It has maximum degree is $\Delta = n$.

An illustration of caterpillar *caterpillar* with 5 internal vertices of degree 3 is presented in Figure 1.

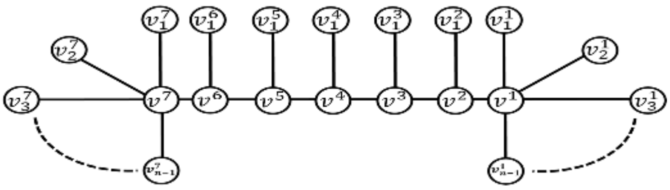

Figure 1. Caterpillar graph $S_{n,\underbrace{3,3,\dots,3}_{5},n}$

The ts of the *caterpillars* as in Definition 3.1.1 is presented in Theorem 3.1.1.

Theorem 3.1.1. If $S_{n,\underbrace{3,3,\dots,3}_{5},n}$, $n \geq 5$ is the *caterpillar* as in Definition 3.1.1, then

$$ts(S_{n,\underbrace{3,3,\dots,3}_{5},n}) = \left\lceil \frac{2n+4}{2} \right\rceil = n+2.$$

Proof. Based on eq. (1):

$$tes(S_{n,\underbrace{3,3,\dots,3}_{5},n}) = max\left\{\left\lceil \frac{|E|+2}{3} \right\rceil, \left\lceil \frac{\Delta+1}{2} \right\rceil\right\} = max\left\{\left\lceil \frac{2n+9+2}{3} \right\rceil, \left\lceil \frac{n+1}{2} \right\rceil\right\} = max\left\{\left\lceil \frac{2n+11}{3} \right\rceil, \left\lceil \frac{n+1}{2} \right\rceil\right\} = \left\lceil \frac{2n+11}{3} \right\rceil$$

Further, based on eq. (2): $tvs(S_{n,\underbrace{3,3,\dots,3}_{5},n}) = \left\lceil \frac{n+1}{2} \right\rceil = \left\lceil \frac{2n+3+1}{2} \right\rceil = \left\lceil \frac{2n+4}{2} \right\rceil$.

The lower bound of ts is obtained from eq. (3):

$$ts\left(S_{n,\underbrace{3,3,\dots,3}_{5},n}\right) \geq max\left\{tes\left(S_{n,\underbrace{3,3,\dots,3}_{5},n}\right), tvs\left(S_{n,\underbrace{3,3,\dots,3}_{5},n}\right)\right\} = max\left\{\left\lceil \frac{2n+11}{3} \right\rceil, \left\lceil \frac{2n+4}{2} \right\rceil\right\}$$
$$= \left\lceil \frac{2n+4}{2} \right\rceil = n+2\,, n \geq 5.$$

Let $k = \left\lceil \frac{2n+4}{2} \right\rceil$. We will prove that $ts(S_{n,\underbrace{3,3,\dots,3}_{5},n}) \leq k$ by constructing a totally irregular total *k*-labelling $h: V \cup E \to \{1,2,\cdots,k\}$ with $k = \left\lceil \frac{2n+4}{2} \right\rceil = n+2$. Meanwhile, vertex and edges-labels are given in Table 1.

Table 1. Labels of elements of the caterpillars with 5 internal vertices

Vertex-labels		Edge-labels	
$h(v_i^1)$	$1, \quad i=1$ $i-1, \forall\, 2 \leq i \leq n-1$	$h(v^1 v_i^1)$	$1, \quad i=1$ $2, \quad \forall\, 2 \leq i \leq n-1$
$h(v_i^7)$	$\left\lceil \frac{2n+4}{2} \right\rceil - 3,\ \forall\, 1 \leq i \leq n-4$ $i+2,\ \forall\, n-3 \leq i \leq n-1$	$h(v^7 v_i^7)$	$i+1, \forall\, 1 \leq i \leq n-4$ $\left\lceil \frac{2n+4}{2} \right\rceil - 4,\ \forall\, n-3 \leq i \leq n-1$
$h(v^j)$	$1, \quad j=1,7$ $\left\lceil \frac{2n+4}{2} \right\rceil + (j-6),\ \forall\, 2 \leq j \leq 6$	$h(v^j v^{j+1})$	$\left\lceil \frac{2n+4}{2} \right\rceil,\ \forall\, 1 \leq j \leq 5$ $\left\lceil \frac{2n+4}{2} \right\rceil - 3,\ j=6$
$h(v_1^j)$	$\left\lceil \frac{2n+4}{2} \right\rceil,\ \forall\, 2 \leq j \leq 6$	$h(v^j v_1^j)$	$\left\lceil \frac{2n+4}{2} \right\rceil + (j-6),\ \forall\, 2 \leq j \leq 6$

We calculate the weights of elements of the caterpillars in Table 2.

Table 2. Weights of elements of the caterpillars with 5 internal vertices

Weights of vertices		Weights of edges	
	$\forall\, 1 \le i \le n-1$		$\forall\, 1 \le i \le n-1$
$w(v^j)$	$3n$, for $j = 1$ $4n + (2j-4), \forall\, 2 \le j \le 5$ $4n + (j-1)$, for $j = 6$ $\frac{n(n+3)}{2} - 4$, for $j = 7$	$w(v^j v^{j+1})$	$2n + 1$, for $j = 1$ $3n + (2j-5), \forall\, 2 \le j \le 5$ $2n + 2$, for $j = 6$
		$w(v^j v_1^j)$	$3n + (2j-6), \forall\, 2 \le j \le 6$
$w(v_i^j)$	$1 + i,\ j = 1$ $n + i,\ j = 7$	$w(v^j v_i^j)$	$2 + i$, for $j = 1$ $1 + n + i$, for $j = 7$
$w(v_1^j)$	$2n + (j-2), \forall\, 2 \le j \le 6$		

It is shown that the labels of elements of the caterpillars are nor more than $k = \left\lceil\frac{2n+4}{2}\right\rceil$. Further, no vertices have a same weight and also all edges have distinct weights. It proves $ts\left(S_{n,\underbrace{3,3,\ldots,3}_{5},n}\right) \le k = \left\lceil\frac{2n+4}{2}\right\rceil$. Thus, $ts(S_{n,\underbrace{3,3,\ldots,3}_{5},n}) = \left\lceil\frac{2n+4}{2}\right\rceil = n + 2$ for $n \ge 5$.

3.2. Caterpillar graphs with 7 internal vertices of degree 3

The notion of a caterpillar with 7 internal vertices of degree 3 is presented in Definition 3.2.1.

Definition 3.2.1. C*aterpillar graphs* $(S_{n,\underbrace{3,3,\ldots,3}_{7},n})$ are obtained from *double star* $S_{n,n}$ by inserting seven vertices $(v^2, v^3, v^4, v^5, v^6, v^7, v^8)$ on the bridge between the two centrals (v^1 and v^9) and the seven vertices are incident to pendant edges $(v_1^j | 2 \le j \le 8\}$. Whereas, the vertices of the double stars are $\{v_i^1 : 1 \le i \le n-1\}$ and $\{v_i^9 : 1 \le i \le n-1\}$. The caterpillar has $2n + 14$ vertices, $2n + 13$ edges, and $2n + 5$ pendant vertices. It has maximum degree is $\Delta = n$.

The ts of the *caterpillars* as in Definition 3.2.1 is provided in Theorem 3.2.1.

Theorem 3.2.1. If $S_{n,\underbrace{3,3,\ldots,3}_{7},n}$, $n \ge 6$ is the *caterpillar* as in Definition 3.2.1, then

$$ts(S_{n,\underbrace{3,3,\ldots,3}_{7},n}) = \left\lceil\frac{2n+6}{2}\right\rceil = n + 3.$$

Proof. The lower bound of *ts* is as follows:

$$ts\left(S_{n,\underbrace{3,3,\ldots,3}_{7},n}\right) \ge max\left\{tes\left(S_{n,\underbrace{3,3,\ldots,3}_{7},n}\right), tvs\left(S_{n,\underbrace{3,3,\ldots,3}_{7},n}\right)\right\} = max\left\{\left\lceil\frac{2n+15}{3}\right\rceil, \left\lceil\frac{2n+6}{2}\right\rceil\right\}$$
$$= \left\lceil\frac{2n+6}{2}\right\rceil = n + 3\ , n \ge 6.$$

Let $k = \left\lceil\frac{2n+6}{2}\right\rceil$. We will prove $ts(S_{n,\underbrace{3,3,\ldots,3}_{7},n}) \le k$ by constructing a totally irregular total *k*-labelling $g : V \cup E \to \{1, 2, \cdots, k\}$ with $k = \left\lceil\frac{2n+6}{2}\right\rceil = n + 3$. Labels of elements of the caterpillar are given in Table 3.

Table 3. Labels of elements of the caterpillars with 7 internal vertices

Vertex-labels		Edge-labels	
$g(v_i^1)$	$1,\quad i = 1$ $i - 1, \forall\, 2 \le i \le n-1$	$g(v^1 v_i^1)$	$1,\quad i = 1$ $2,\quad \forall\, 2 \le i \le n-1$

$g(v_i^9)$	$\left\lceil\frac{2n+6}{2}\right\rceil - 5,\ \forall\, 1 \le i \le n-5$ $i+3,\ \forall\, n-4 \le i \le n-1$	$g(v^9 v_i^9)$	$i+2,\ \forall\, 1 \le i \le n-5$ $\left\lceil\frac{2n+6}{2}\right\rceil - 6,\ \forall\, n-4 \le i \le n-1$
$g(v^j)$	$1,\ j = 1,9$ $\left\lceil\frac{2n+6}{2}\right\rceil + (j-8),\ \forall\, 2 \le j \le 8$	$g(v^j v^{j+1})$	$\left\lceil\frac{2n+6}{2}\right\rceil,\ \forall\, 1 \le j \le 7$ $\left\lceil\frac{2n+6}{2}\right\rceil - 5,\ j = 8$
$g(v_1^j)$	$\left\lceil\frac{2n+6}{2}\right\rceil,\ \forall\, 2 \le j \le 8$	$g(v^j v_1^j)$	$\left\lceil\frac{2n+6}{2}\right\rceil + (j-8),\ \forall\, 2 \le j \le 8$

It is obvious that the labels of elements of the caterpillar is not more than $k = \left\lceil\frac{2n+6}{2}\right\rceil$. Moreover, we evaluate the weights in Table 4.

Table 4. Weights of elements of the caterpillars with 7 internal vertices

Weights of vertices		**Weights of edges**	
	$\forall\, 1 \le i \le n-1$		$\forall\, 1 \le i \le n-1$
$w(v^j)$	$3n+1,\ \text{for } j = 1$ $4n+(2j-4),\ \forall\, 2 \le j \le 7$ $4n+(j-1),\ \text{for } j = 8$ $\frac{n(n+5)}{2} - 13,\ j = 9$	$w(v^j v^{j+1})$	$2n+1,\ \text{for } j = 1$ $3n+(2j-6),\ \forall\, 2 \le j \le 7$ $2n+2,\ \text{for } j = 8$
		$w(v^j v_1^j)$	$3n+(2j-7),\ \forall\, 2 \le j \le 8$
$w(v_i^j)$	$1+i,\ j = 1$ $n+i,\ j = 9$	$w(v^j v_i^j)$	$2+i,\ j = 1$ $1+n+i, j = 9$
$w(v_1^j)$	$2n+(j-2),\ \forall\, 2 \le j \le 8$		

Based on the above calculation, we can see that the vertices have different weights and no edges have a same weight. Therefore, the upper bound is obtained and $ts(S_{n,\underbrace{3,3,\ldots,3}_{7},n}) = \left\lceil\frac{2n+6}{2}\right\rceil = n+3$ for $n \ge 6$. An illustration of labelling on the caterpillar $S_{n,\underbrace{3,3,\ldots,3}_{7},n}$ is shown in Figure 2. The green colors show vertex-labels and the blue colors denote labels of edges.

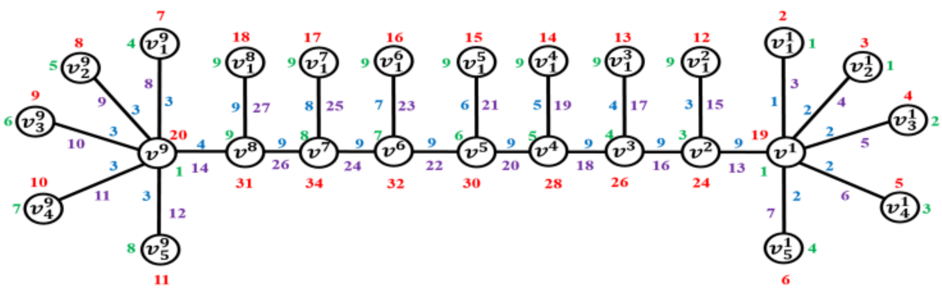

Figure 2. The totally irregular total 9-labelling on the *Caterpillar* $S_{n,\underbrace{3,3,\ldots,3}_{7},n}$

3.3. Caterpillar graphs with 9 internal vertices of degree 3

Definition 3.3.1. *Caterpillar* $S_{n,\underbrace{3,3,\ldots,3}_{9},n}$ is a graph which is obtained from *double star* $S_{n,n}$ by inserting nine vertices $(v^2, v^3, v^4, v^5, v^6, v^7, v^8, v^9, v^{10})$ on the bridge connecting the two centres (v^1 and v^{11}) and the nine vertices are incident to pendant edges $(v_1^j | 2 \le j \le 10\}$. Meanwhile, the vertices of the double stars are $\{v_i^1 : 1 \le i \le n-1\}$ and $\{v_i^{11} : 1 \le i \le n-1\}$. The caterpillar has $2n+18$ vertices, $2n+17$ edges, and $2n+7$ pendant vertices. Its maximum degree is $\Delta = n$.

The ts of the *caterpillars* that contain nine internal vertices of degree 3 is proved in Theorem 3.3.1.

Theorem 3.3.1. If $S_{n,\underbrace{3,3,\ldots,3}_{9},n}$, $n \geq 7$ is the *caterpillar* as in Definition 3.3.1, then

$$ts(S_{n,\underbrace{3,3,\ldots,3}_{9},n}) = \left\lceil \frac{2n+8}{2} \right\rceil = n+4.$$

Proof. It is similar to the proof of Theorem 3.1.1 and Theorem 3.2.1, we get a lower bound as follows:

$$ts(S_{n,\underbrace{3,3,\ldots,3}_{9},n}) \geq max\left\{tes(S_{n,\underbrace{3,3,\ldots,3}_{9},n}), tvs(S_{n,\underbrace{3,3,\ldots,3}_{9},n})\right\} = max\left\{\left\lceil \frac{2n+19}{3} \right\rceil, \left\lceil \frac{2n+8}{2} \right\rceil\right\} = \left\lceil \frac{2n+8}{2} \right\rceil$$
$$= n+4, n \geq 7.$$

Let $k = \left\lceil \frac{2n+8}{2} \right\rceil$. We should show $ts(S_{n,\underbrace{3,3,\ldots,3}_{9},n}) \leq k$ by constructing a totally irregular total k-labelling $p: V \cup E \to \{1,2,\cdots,k\}$ with $k = \left\lceil \frac{2n+8}{2} \right\rceil = n+4$. We define labels for elements of the caterpillar to Table 5.

Table 5. Labels of vertices and edges in the caterpillars with 9 internal vertices

$p(v)$ **For all** $v \in V(G)$		$p(e)$ **For all** $e \in E(G)$	
$p(v_i^1)$	$1, \quad i = 1$ $i-1, \ \forall\, 2 \leq i \leq n-1$	$p(v^1 v_i^1)$	$1, \quad i = 1$ $2, \ \forall\, 2 \leq i \leq n-1$
$p(v_i^{11})$	$\left\lceil \frac{2n+8}{2} \right\rceil - 7, \ \forall\, 1 \leq i \leq n-6$ $i+4, \ \forall\, n-5 \leq i \leq n-1$	$p(v^{11} v_i^{11})$	$i+3, \ \forall\, 1 \leq i \leq n-6$ $\left\lceil \frac{2n+8}{2} \right\rceil - 8, \ \forall\, n-5 \leq i \leq n-1$
$p(v^j)$	$1, \quad j = 1,11$ $\left\lceil \frac{2n+8}{2} \right\rceil + (j-10), \ \forall\, 2 \leq j \leq 10$	$p(v^j v^{j+1})$	$\left\lceil \frac{2n+8}{2} \right\rceil, \ \forall\, 1 \leq j \leq 9$ $\left\lceil \frac{2n+8}{2} \right\rceil - 7, \ j = 10$
$p(v_1^j)$	$\left\lceil \frac{2n+8}{2} \right\rceil, \ \forall\, 2 \leq j \leq 10$	$p(v^j v_1^j)$	$\left\lceil \frac{2n+8}{2} \right\rceil + (j-10), \ \forall\, 2 \leq j \leq 10$

It is shown above that the labels of elements of the caterpillar is less than or equal to $k = \left\lceil \frac{2n+8}{2} \right\rceil$. Furthermore, we calculate the weights to Table 6.

Table 6. Weights of vertices and edges in the caterpillars with 9 internal vertices

$w(v)$ **For all** $v \in V(G)$		$w(e)$ **For all** $e \in E(G)$	
	$\forall\, 1 \leq i \leq n-1$		$\forall\, 1 \leq i \leq n-1$
$w(v^j)$	$3n+2, \ j = 1$ $4n+(2j-4), \ \forall\, 2 \leq j \leq 9$ $4n+(j-1), \ j = 10$ $\frac{n(n+7)}{2} - 25, \ j = 11$	$w(v^j v^{j+1})$	$2n+1, \ j = 1$ $3n+(2j-7), \ \forall\, 2 \leq j \leq 9$ $2n+2, \ j = 10$
		$w(v^j v_1^j)$	$3n+(2j-8), \ \forall\, 2 \leq j \leq 9$
$w(v_i^j)$	$1+i, \quad j = 1$ $n+i, \quad j = 11$	$w(v^j v_i^j)$	$2+i, \ j = 1$ $1+n+i, \ j = 11$
$w(v_1^j)$	$2n+(j-2), \ \forall\, 2 \leq j \leq 10$		

We observe that all elements of the caterpillar do not have a same weight. Therefore, we get the upper bound and $ts(S_{n,\underbrace{3,3,\ldots,3}_{9},n}) = \left\lceil \frac{2n+8}{2} \right\rceil = n+4$ for $n \geq 7$.

Figure 3 describes labelling on the *caterpillar* $S_{n,\underbrace{3,3,\ldots,3}_{9},n}$. The green colors indicate labels of vertices and the blue colors present edge-labels.

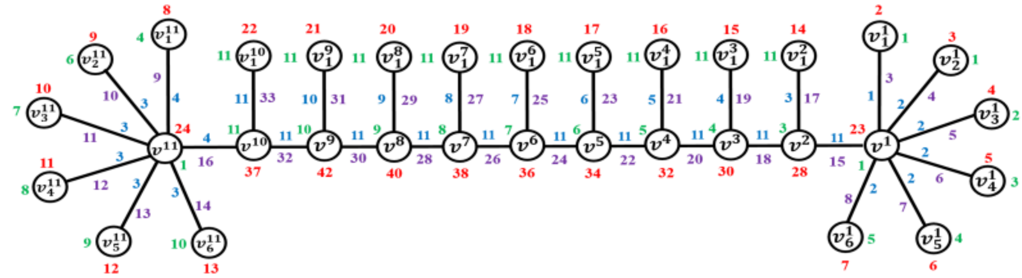

Figure 3. The totally irregular total 11-labelling on the *Caterpillar* $S_{n,\underbrace{3,3,\ldots,3}_{9},n}$

4. Conclusion

In this research, we proved that ts of $(S_{n,\underbrace{3,3,\ldots,3}_{q},n})$ is equal to: $\left\lceil\frac{2n+4}{2}\right\rceil = n+2$ for q=5, it is equal to $\left\lceil\frac{2n+6}{2}\right\rceil = n+3$ for q=6, and it is equal to $\left\lceil\frac{2n+8}{2}\right\rceil = n+4$ for q=9. In upcoming research, we will investigate ts of the caterpillars which have q internal vertices of degree 3 for odd $q > 9$.

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