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On total edge irregularity strength of tadpole chain graph $\boldsymbol{T_r(6,n)}$

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Abstract. Given a graph $G(V,E)$ with a non-empty set of vertices V and a setof edges E. A total labelling $f: V \cup E \rightarrow \{1,2,\dots,k\}$ is called an edge irregular total labeling if the weight of every edge is distinct. The weight of anedgee, under the total labeling f, is the sum of label of edge$e$ and all labels of vertices that are incident to e. In other words, $wt(xy) = f(xy) + f(x) + f(y)$. The total edge irregularity strength of G, denoted by $tes(G)$ is the minimum k used to label graph G with the edge irregular total labeling. A tadpole chain graph of length r, denoted as $T_r(6,n)$, is a chain graph that consists of tadpole graph $T(6,n)$ on each block. In this paper, we get $tes(T_r\ (6,n)\ ) = \left\lceil\frac{(6+n)r+2}{3}\right\rceil$ and construct an algorithm to find it.

1. Introduction

Given a simple, connected and undirected graph $G = (V(G), E(G))$. A labelling of G is a function that assigns a set of elements of G into a set of positive integers [14]. A labelling $f$on$G$is said to be a total labeling if its domain is union $V(G) \cup E(G)$.Bača et al. [3] defined an edge irregular total k-labeling as a function $f : V(G) \cup E(G) \rightarrow \{1,2,\dots,k\}$which has the weights $wt(uv) \neq wt(xy)$ for every two different edges uv and xy, where $wt(e) = wt(uv) = f(u) + f(uv) + f(v)$.Further, a total edge irregularity strength of G,symbolized by $tes(G)$, is a minimum number$k$ in edge irregular total k-labeling.

The bounds for tes of any graph G was given by Bača et al. [3] as the following:

$$\left\lceil\frac{|E(G)|+2}{3}\right\rceil \leq tes(G) \leq |E|. \tag{1}$$

Meanwhile, Ivančo and Jendrol [8] found a conjecture for tes of graph$G$:

$$tes(G) = max\left\{\left\lceil\frac{|E(G)|+2}{3}\right\rceil, \left\lceil\frac{\Delta(G)+1}{2}\right\rceil\right\} \tag{2}$$

where $\Delta(G)$ is a maximum degree of all vertices of G.

The proof of Conjecture (2) has been revealed by some researchers for some special graphs, such as: Jendrol et al. [9] verifiedtes of complete and complete bipartite graphs; Ivančo and Jendrol [8] gave tes of any tree. Furthermore, tes of some graph classes has been investigated by many researches as well as presented in Gallian [4]. Mushayt and Ahmad investigated a$tes$of hexagonal grid graphs [10]. Indriati et al. ([6],[7]) found tes of generalized helm and generalized web graphs. Nurdini and Rosyida [11] found tes of dovetail graph with some pendant vertices and related graph. Rosyida and Indriati [13] provided tes of C_3 and C_4 cactus chain graphs with pendant vertices. The readers can find more results on es, tes, and tvs of graphs in [5].

The authors were encouraged by the results in Rosyida et al. [12] that determined$tvs$of $T_r(4,1)$ tadpolechain graph. The authors were also interested to the result in [5] that gave an edge irregularity strength (es) of cycle chain graphs and results from Ahmad et al. [1] which proposed es of several chain graphs. The problem investigated in this paper is different to the result in [12], [5] and [1]. We verify tes of tadpole chain graph $T_r(6,n)$ and construct an algorithm to find it.

2. Main Results

In the following investigation, we discuss an exact value of the total edge irregularity strength of $T_r(6,n)$tadpole chain graph as presented in Theorem 2.1. We refer the concept of $T_r(6,n)$tadpole chain graph from [2] and [12].

Definition 1 *A tadpole graph$T_{(k,n)}$, is the graph created by concatenating an edge from any vertex of$C_k$with a pendant of $P_n$for integers$k \geq 3$and$n \geq 1$.The tadpole graph contains $m+n$ vertices and $m+n$ edges.*

Definition 2*Givena connected graph $G(V,E)$. A block cut vertex graph of G is a graph in which the vertices are the blocks and cut vertices of$G$. A chain graph is a graph which contains some blocks $B_1, B_2, \dots, B_r$so that each pair of blockB_i, B_{i+1}has at most one common cut vertex such that the block cut vertex is a path. A chain graph which each block is tadpole graph is called tadpole chain graph.*

In a $T_r(6,n)$tadpole chain graph, each hexagon has cut vertices at most two, each of two hexagons has one common cut vertex, and a path $P_n$concatenated in each hexagon. The length of the chain is indicated by the number r on$T_r(6,n)$tadpole chain graph. The notation $T_r(6,n)$ stands for a$T(6,n)$tadpole chain graph with length r. The formula for tes of $T_r(6,n)$ is presented in Theorem 2.3.

Theorem1 *Given a tadpole chain graph $T_r(6,n)$ with length r and P_n concatenated in each hexagon. The total edge irregularity strength of $T_r(6,n)$ is*

$$tes\left(T_r(6,n)\right) = \left\lceil \frac{(6+n)r+2}{3} \right\rceil.$$

Proof. Tadpole chain graph $T_r(6,n)$consists of $(6+n)r$ edges. Let $u_{2i-1,}, u_{2i}, v_i, x_{2i-1}, x_{2i}$ be vertices located on each hexagon. Let $u_{2i-1,}, u_{2i}$ be the two vertices on the top of hexagon for $i = 1,2,\dots,2r$, let v_i be the cut vertices for $i = 1,2,\dots,r$, and x_{2i-1}, x_{2i} be vertices located on the bottom of hexagon for $i = 1,2,\dots,2r$. Let x_{2i-1} be the vertice that concatenated with y_i^n which is the part of y_i^j for $i = 1,2,\dots,r$ and $j = 1,2,\dots,n$. The lower bound for tes of the graph $T_r(6,n)$ is as follows [3]:

$$\left\lceil \frac{(6+n)r+2}{3} \right\rceil \leq tes\left(T_r(6,n)\right) \leq (6+n)r.$$

Further, we show the upper bound of $\text{tes}\left(T_r(6,n)\right) \leq \left\lceil \frac{(6+n)r+2}{3} \right\rceil$by constructing a total k-labeling $f: V \cup E \rightarrow \{1,2,\dots,k\}$ where $k = \left\lceil \frac{(6+n)r+2}{3} \right\rceil$ as follows.

Case 1. For $n = 3 \ mod\ 3$:

Labels of vertices are defined as the following:

$$f(U_{2i-1}) = f(U_{2i}) = \left\lceil \frac{(6+n)i+2}{3} \right\rceil, i = 1,2,\dots,r$$

$$f(X_{2i-1}) = f(X_{2i}) = \left\lceil \frac{(6+n)i+2}{3} \right\rceil - 1, i = 1,2,\dots,r$$

$$f(V_i) = \left\lceil \frac{(6+n)i+2}{3} \right\rceil - 3, i = 1,2,\dots,r$$

$$f\left(Y_i^j\right) = \frac{(6+n)i - n + \left(3\left\lceil\frac{j}{3}\right\rceil - 6\right)}{3}, i = 1,2,\dots,r;\ j = 1,2,\dots,n$$

Meanwhile, labels of edges are:

$$f(U_{2i-1}U_{2i}) = \left\lceil\frac{(6+n)i+2}{3}\right\rceil - 3, i = 1,2,\dots,r$$

$$f(U_{2i-1}V_i) = f(V_iX_{2i-1}) = f(X_{2i-1}X_{2i}) = \left\lceil\frac{(6+n)i+2}{3}\right\rceil - 2, i = 1,2,\dots,r$$

$$f(U_{2i-2}V_i) = f(V_iX_{2i-2}) = \frac{(6+n)i-n+3}{3}, i = 1,2,\dots,r$$

$$f\left(Y_i^jY_i^{j+1}\right) = \frac{(6+n)i - n + \left(3\left\lceil\frac{j+2}{3}\right\rceil - 6\right)}{3}, i = 1,2,\dots,r;\ j = 1,2,\dots,n$$

$$f(X_{2i-1}Y_i^n) = \left\lceil\frac{(6+n)i-8}{3}\right\rceil - 3, i = 1,2,\dots,r$$

Case 2. For $n \neq 3\ mod\ 3$:

Labels of vertices are defined as follows:

$$f(U_{2i-1}) = f(U_{2i}) = \begin{cases} 3, if\ i = 1; n = 2 \\ \left\lceil\frac{(6+n)i+2}{3}\right\rceil, i = 1,2,\dots,r \end{cases}$$

$$f(X_{2i-1}) = \begin{cases} 1, if\ i = 1; n = 1 \\ 2, if\ i = 1; n = 2 \\ \left\lceil\frac{(6+n)i+2}{3}\right\rceil - 1, i = 1,2,\dots,r \end{cases}$$

$$f(X_{2i}) = \begin{cases} 2, if\ i = 1; n = 2 \\ \left\lceil\frac{(6+n)i+2}{3}\right\rceil - 1, i = 1,2,\dots,r \end{cases}$$

$$f(V_i) = \begin{cases} 1, if\ i = 1\ and\ n = 1 \\ \left\lceil\frac{(6+n)i+2}{3}\right\rceil - 3, i = 1,2,\dots,r \end{cases}$$

$$f\left(Y_i^j\right) = \left\lceil\frac{(6+n)i-n+j-6}{3}\right\rceil, i = 1,2,\dots,r;\ j = 1,2,\dots,n$$

Meanwhile, labels of edges are:

$$f(U_{2i-1}U_{2i}) = \begin{cases} 2, if\ i = 1, n = 2 \\ \left\lceil \frac{(6+n)i+2}{3} \right\rceil - 4, & if\ ((6+n)i)\ mod\ (3) = 2, i = 1,2,\dots,r \\ \left\lceil \frac{(6+n)i+2}{3} \right\rceil - 2, & if\ ((6+n)i)\ mod\ (3) = 1, i = 1,2,\dots,r \\ \left\lceil \frac{(6+n)i+2}{3} \right\rceil - 3, & if\ ((6+n)i)\ mod\ (3) = 0, i = 1,2,\dots,r \end{cases}$$

$$f(U_{2i-1}V_i) = \begin{cases} 1, if\ i = 1, n = 1 \\ 2, if\ i = 1, n = 2 \\ \left\lceil \frac{(6+n)i+2}{3} \right\rceil - 3, & if\ ((6+n)i)\ mod\ (3) = 2, i = 1,2,\dots,r \\ \left\lceil \frac{(6+n)i+2}{3} \right\rceil - 1, & if\ ((6+n)i)\ mod\ (3) = 1, i = 1,2,\dots,r \\ \left\lceil \frac{(6+n)i+2}{3} \right\rceil - 2, & if\ ((6+n)i)\ mod\ (3) = 0, i = 1,2,\dots,r \end{cases}$$

$$f(V_iX_{2i-1}) = \begin{cases} 2, if\ i = 1, n = 2 \\ \left\lceil \frac{(6+n)i+2}{3} \right\rceil - 3, & if\ ((6+n)i)\ mod\ (3) = 2, i = 1,2,\dots,r \\ \left\lceil \frac{(6+n)i+2}{3} \right\rceil - 1, & if\ ((6+n)i)\ mod\ (3) = 1, i = 1,2,\dots,r \\ \left\lceil \frac{(6+n)i+2}{3} \right\rceil - 2, & if\ ((6+n)i)\ mod\ (3) = 0, i = 1,2,\dots,r \end{cases}$$

$$f(X_{2i-1}X_{2i}) = \begin{cases} 3, if\ i = 1, n = 1,2 \\ \left\lceil \frac{(6+n)i+2}{3} \right\rceil - 3, & if\ ((6+n)i)\ mod\ (3) = 2, i = 1,2,\dots,r \\ \left\lceil \frac{(6+n)i+2}{3} \right\rceil - 1, & if\ ((6+n)i)\ mod\ (3) = 1, i = 1,2,\dots,r \\ \left\lceil \frac{(6+n)i+2}{3} \right\rceil - 2, & if\ ((6+n)i)\ mod\ (3) = 0, i = 1,2,\dots,r \end{cases}$$

$$f(U_{2i-2}V_i) = f(V_iX_{2i-2}) = \begin{cases} 4, if\ i = 1\ and\ n = 2 \\ \left\lceil \frac{(6+n)i-(n-2)}{3} \right\rceil, i = 1,2,\dots,r \end{cases}$$

$$f\left(Y_i^j Y_i^{j+1}\right) = \left\lceil \frac{(6+n)i-n+j-4}{3} \right\rceil, i = 1,2,\dots,r;\ j = 1,2,\dots,n$$

$$f(X_{2i-1}Y_i^n) = \begin{cases} 1, i = 1, n = 1,2 \\ \left\lceil \frac{(6+n)i-8}{3} \right\rceil, i = 1,2,\dots,r \end{cases}$$

Since we get the labels of vertices and edges are less than or equal to$k = \left\lceil \frac{(6+n)r+2}{3} \right\rceil$, then the labeling f is ak-total labeling.

Further, we verify that the weights of edges are distinct under the function f as follows:
$wt(U_{2i-1}U_{2i}) = (6+n)i, i = 1,2,\dots,r$
$wt(U_{2i-1}V_i) = (6+n)i - 2, i = 1,2,\dots,r$
$wt(V_iX_{2i-1}) = (6+n)i - 3, i = 1,2,\dots,r$
$wt(X_{2i-1}X_{2i}) = (6+n)i - 1, i = 1,2,\dots,r$

$wt(U_{2i-2}V_i) = (6+n)i + 2, i = 1,2,\dots,r$
$wt(V_iX_{2i-2}) = (6+n)i + 1, i = 1,2,\dots,r$
$wt\left(Y_i^jY_i^{j+1}\right) = (6+n)i - n + j - 4, i = 1,2,\dots,r, j = 1,2,\dots,n$
$wt(X_{2i-1}Y_i^n) = wt\left(Y_i^{n-1}Y_i^n\right) + 1, i = 1,2,\dots,r$

It is clear that the weights of all edges are distinct and we obtain upper bound$tes\left(T_r(6,n)\right) \le \left\lceil\frac{(6+n)r+2}{3}\right\rceil$. Thus, we show that tes of tadpole chain graph $T_r(6,n)$ as follows:

$$tes\left(T_r(6,n)\right) = \left\lceil\frac{(6+n)r+2}{3}\right\rceil.$$

3. Computational results

In this section, we present computational result of tes of $T_r(6,n)$ graph. Acomputer program by using Matlab R2016a is constructed based on an algorithm in Table 1.

Table1. Algorithm to determine tes of $T_r(6,n)$ tadpole chain graph.

Commands:
Input r %Length of chain graph
input n %Number ofvertices in path Pn
for i=1 to r % assign labels to vertices of G
$f(U_{2i-1}) = \text{ceil}(\frac{(6+n)i+2}{3})$; $f(U_{2i}) = \text{ceil}(\frac{(6+n)i+2}{3})$
$f(X_{2i-1}) = \text{ceil}(\frac{(6+n)i+2}{3}) - 1$; $f(X_{2i}) = \text{ceil}(\frac{(6+n)i+2}{3}) - 1$
$$f(V_i) = \text{ceil}\left(\frac{(6+n)i+2}{3}\right) - 3$$
for j=1 to n
$$f\left(Y_i^j\right) = \frac{(6+n)i - n + \left(3ceil\left(\frac{j}{3}\right) - 6\right)}{3}$$
end
end
for i=1 to r % assign labels to edges and determine the weights
$f(U_{2i-1}U_{2i}) = \text{ceil}\left(\frac{(6+n)i+2}{3}\right) - 3$;
$$wt(U_{2i-1}U_{2i}) = (6+n)i;$$
$$f(U_{2i-1}V_i) = \text{ceil}\left(\frac{(6+n)i+2}{3}\right) - 2$$
$$wt(U_{2i-1}V_i) = (6+n)i - 2$$
$$f(V_iX_{2i-1}) = \text{ceil}\left(\frac{(6+n)i+2}{3}\right) - 2$$
$$wt(V_iX_{2i-1}) = (6+n)i - 3$$
$$f(X_{2i-1}X_{2i}) = \text{ceil}\left(\frac{(6+n)i+2}{3}\right) - 2$$
$$wt(X_{2i-1}X_{2i}) = (6+n)i - 1$$

Commands:

$$f(U_{2i}V_{i+1}) = \frac{(6+n)i-n+3}{3}$$

$$wt(U_{2i}V_{i+1}) = (6+n)i+2,$$

$$f(V_{i+1}X_{2i}) = \frac{(6+n)i-n+3}{3}$$

$$wt(V_{i+1}X_{2i}) = (6+n)i+1,$$

$$f(X_{2i-1}Y_i^n) = \text{ceil}\left(\frac{(6+n)i-8}{3}\right)-3$$

```
for j=1 to n-1
```

$$(Y_i^j Y_i^{j+1}) = \frac{(6+n)i-n+\left(3\text{ceil}\left(\frac{j+2}{3}\right)-6\right)}{3}$$

$$wt(Y_i^j Y_i^{j+1}) = (6+n)i-n+j-4$$

```
end
end
Print [f(edges) wt(edges)]%display edges and the weighs
```

As a simulation, we give an illustration of the edge irregular total 13-labeling of $T_4(6,3)$ in Figure 1. The weight of each edge is printed in the red color. By using the algorithm, the labeling output and tes of $T_r(6,3)$ from computer program is given in Figure 2.

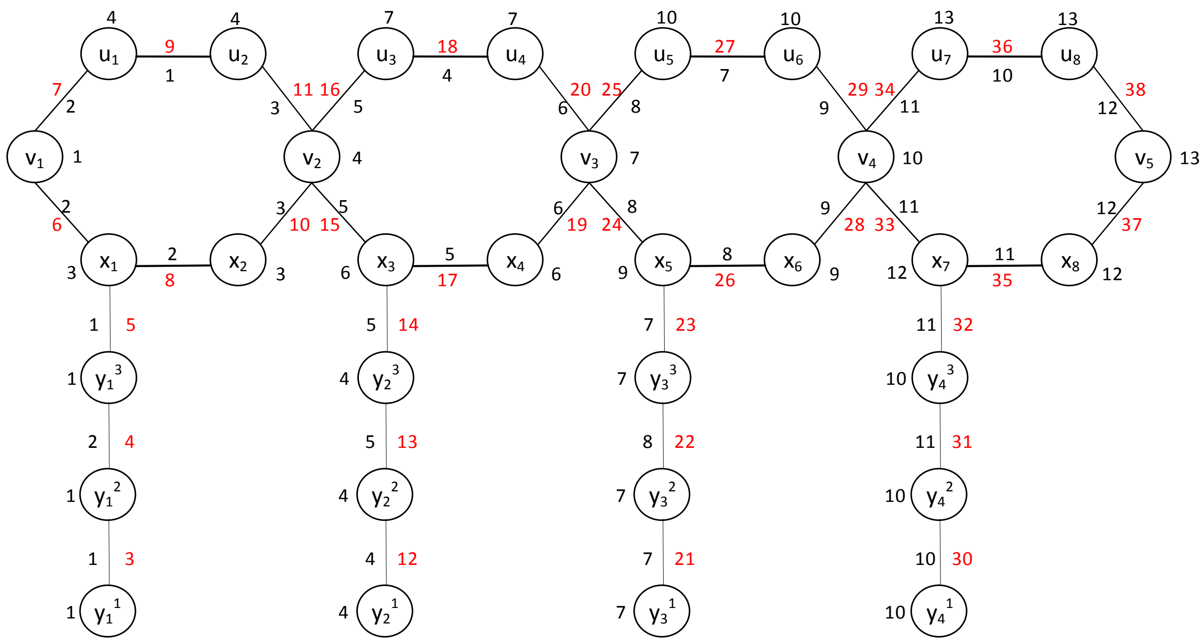

Figure 1. The total 13-labeling of $T_4(6,3)$.

Matlab output for determining tes of $T_4(6,3)$ is presented in Figure 2.

Label_u(2i)	Label_x(2i)
0	0
4	3
0	0
7	6
0	0
10	9
0	0
13	12

Label_y1	Label_y2	Label_y3	Label_y4
1	4	7	10
1	4	7	10
1	4	7	10

Label_v
1
4
7
10
13

Edges_u(2i-1) u(2i)		Label	Weight
1	2	1	9
3	4	4	18
5	6	7	27
7	8	10	36

Edges_u(2i-1) v(i)		Label	Weight
1	1	2	7
3	2	5	16
5	3	8	25
7	4	11	34

Edges_x(2i-1) x(2i)		Label	Weight
1	2	2	8
3	4	5	17
5	6	8	26
7	8	11	35

Edges_x(2i) v(i)		Label	Weight
2	2	3	10
4	3	6	19
6	4	9	28
8	5	12	37

Edges_x(2i-1) y(i)		Label	Weight
1	1	1	5
3	2	4	14
5	3	7	23
7	4	10	32

Edges_y1_y2		Label	Weight
1	2	1	3
2	3	2	4

Edges_y5_y6		Label	Weight
1	2	7	21
2	3	8	22

Edges_y5_y6		Label	Weight
1	2	7	21
2	3	8	22

Figure 2. The labeling output of total 13-labeling of $T_4(6,3)$ by Matlab.

4. Conclusions

In this paper, we have invented and proved tes oftadpole chain graph$T_r(6,n)$.We found that $tes\left(T_r(6,n)\right) = \left\lceil\frac{(6+n)r+2}{3}\right\rceil$ and an algorithm to find the tes is also constructed. In upcoming work, we will investigate tes of generalized tadpole chain and generalized cactus chain graphs. Moreover, we present an open problem for further research.

Open Problems The total vertex irregularity strength of generalized tadpole chain and generalized cactus chain graphs

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