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Procedia Computer Science 157 (2019) 699–706

Procedia Computer Science

www.elsevier.com/locate/procedia

4th International Conference on Computer Science and Computational Intelligence 2019 (ICCSCI), 12-13 September 2019

Determining Total Vertex Irregularity Strength of $T_r(4, 1)$ Tadpole Chain Graph and its Computation

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Abstract

In this research, we examine total vertex irregularity strength (tvs) of tadpole chain graph $T_r(4, 1)$ with length r. We obtain that tvs $(T_r(4, 1)) = \left\lceil \frac{4r + 2}{5} \right\rceil$. Further, we construct algorithm to determine label of vertices, label of edges, weight of vertices, and the exact value of tvs of $T_r(4, 1)$.

Peer-review under responsibility of the scientific committee of the 4th International Conference on Computer Science and Computational Intelligence 2019.

Keywords: Vertex weight; tvs; tadpole; cactus chain graph.

1. Introduction

Let $G(V, E)$ be a simple, finite, and undirected graph. A function f is named *total labeling* if it is map $V \cup E$ into a set of integers [1] which are called labels. A function $f : V \cup E \rightarrow \{1, 2, \ldots, k\}$ is called a vertex irregular total k-labeling of G if the vertex weights $wt_f(u) \neq wt_f(v)$ for all diverse vertices in G with $w_f(u) = f(u) + \sum_{ux \in E(G)} f(ux)$. The concept of total vertex irregularity strength of graph G was introduced by Baca et al. [2] that is the minimum number $k$such that G has a vertex irregular total k-labeling, denoted as tvs(G).

Further, the function f is named an edge irregular total k-labeling of G if the weights $wt_f(uv) \neq wt_f(xy)$ for every two different edges uv and xy in $E(G)$, with $wt_f(uv) = f(u) + f(v) + f(uv)$. The minimum number k is called a total edge irregularity strength of G, denoted by tes(G) [2].

The bounds for tvs of graphs are as follows [3]:

$$\left\lceil \frac{(p+\delta)}{(\Delta+1)} \right\rceil \leq tvs(G) \leq p + \Delta - 2\delta + 1,$$

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Peer-review under responsibility of the scientific committee of the 4th International Conference on Computer Science and Computational Intelligence 2019.
10.1016/j.procs.2019.09.152

where p is the number of vertices of G, δ and Δ are minimum and maximum degree of all vertices of G, respectively. Another bound was given by Anholcer, et al.[4]. Meanwhile, Nurdin et al.[5] gave a lower bound of tvs for any connected graph G having n_i vertices of degree i with $i = \delta, \delta + 1, \delta + 2, \ldots, \Delta$ as follows:

$$tvs(G) \geq \max\left\{\left\lceil \frac{\delta(G) + n_{\delta(G)}}{\delta(G) + 1} \right\rceil, \left\lceil \frac{\delta(G) + n_{\delta(G)} + n_{\delta(G)+1}}{\delta(G) + 2} \right\rceil, \ldots, \left\lceil \frac{\delta(G) + \sum_{i=\delta(G)}^{\Delta(G)} n_i}{\Delta(G) + 1} \right\rceil\right\} \tag{1}$$

Some exact values of total vertex irregularity strength of any class graphs have been found by researchers. Readers are referred to[2],[6],[7],[8],[9],[10],[11],[3], etc. This research is motivated by the result in[12]. In this paper, we continue the result in[12] and[13] that is investigating tvs of tadpole chain graph $T_r(4, 1)$ and construct a related algorithm.

Nomenclature

tvs	total vertex irregularity strength
$T(4, 1)$	tadpole graph which is formed of cycle C_4 and path P_1 connected by a bridge
$T_r(4, 1)$	tadpole cactus chain graph of length r

2. Main Results

In subsection 1, the result of tvs of tadpole chain graph $T_r(4, 1)$ is presented. Further, computational result in determining tvs of $T_r(4, 1)$ is given in subsection 2.

2.1. *Formula for tvs of tadpole chain graph $T_r(4, 1)$*

Firstly, a concept of tadpole graph[14] and tadpole chain graph are given as follows.

Definition 1. *A tadpole graph $T(m, n)$ is graph consisting of a cycle graph of m vertices and a path graph of n vertices connected with a bridge. Therefore, a tadpole graph consists of $m + n$ vertices and $m + n$ edges.*

In this article, we use a tadpole graph $T(4, 1)$.

Definition 2. *The block cut vertex graph of a connected graph G is the graph whose vertices are the blocks and cut vertices of G. A chain graph is a connected graph which contains some blocks $B_1, B_2, \ldots, B_r$ such that every two blocks B_i, B_{i+1} have at most one common cut vertex in such away that the block cut vertex is a path[12]. The tadpole chain graph is a chain graph with all blocks are tadpole graphs.*

In this article, we focus on tadpole chain graph which all blocks are tadpole $T(4, 1)$. The length of tadpole chain graph is indicated by the number of blocks in the chain. Tadpole chain graph with length r is denoted as $T_r(4, 1)$. A formula for tvs of $T_r(4, 1)$ is presented in Theorem 1.

Theorem 1. *Let $T_r(4, 1)$ be a tadpole chain graph with length r ($r \geq 3$). The total vertex irregularity strength of $T_r(4, 1)$ is*

$$tvs(T_r(4, 1)) = \left\lceil \frac{4r + 2}{5} \right\rceil.$$

***Proof*.** Tadpole cactus chain graphs $T_r(4, 1)$ with length r contain r vertices which have degrees 1, $r+2$ vertices which have degrees 2, r vertices with degrees 3, and $r - 1$ vertices with degrees 4. We denote that $y_1, y_2, \ldots, y_r$ are vertices of degree 1; $u_1, u_2, \ldots, u_r, u_{r+1}, u_{r+2}$ are vertices of degree 2; $x_1, x_2, \ldots, x_r$ are vertices of degree 3; and $v_1, v_2, \ldots, v_{r-1}$ are vertices of degree 4. According to inequality (1), we obtain

$$tvs(T_r(4,1)) \geq \max\left\{\left\lceil\frac{r+1}{2}\right\rceil, \left\lceil\frac{2r+3}{3}\right\rceil, \left\lceil\frac{3r+3}{4}\right\rceil, \left\lceil\frac{4r+2}{5}\right\rceil\right\} = \left\lceil\frac{4r+2}{5}\right\rceil \quad (2)$$

To obtain an upper bound for $tvs(T_r(4,1))$, we verify the existence of a vertex iregular total k-labeling of $T_r(4,1)$. Therefore, we construct a function f from $V \cup E$ to $\{1, 2, \ldots, k\}$ with $k = \left\lceil\frac{4r+2}{5}\right\rceil$ in the following three cases.

Case 1. For $r \equiv 0 \mod \left\lceil\frac{4r+2}{5}\right\rceil$ $(3 \leq r \leq 6)$.

Label of vertices are defined as:

$$\begin{aligned}
f(u_i) &= 1,\ 1 \leq i \leq r,\\
f(u_{r+1}) &= 2,\\
f(u_{r+2}) &= 3,\\
f(v_i) &= 3,\ 1 \leq i \leq r-1,\\
f(x_i) &= 3,\ 1 \leq i \leq r,\\
f(y_i) &= 1,\ 1 \leq i \leq r.
\end{aligned}$$

Meanwhile, label of edges are:

$$\begin{aligned}
f(u_1u_{r+1}) &= \left\lceil\tfrac{4r+2}{5}\right\rceil,\\
f(u_ru_{r+2}) &= \left\lceil\tfrac{4r+2}{5}\right\rceil,\\
f(u_iv_i) &= i,\ 1 \leq i \leq r-1,\\
f(u_{i+1}v_i) &= \left\lceil\tfrac{4r+2}{5}\right\rceil,\ 1 \leq i \leq r-1,\\
f(v_ix_i) &= \left\lceil\tfrac{4r+2}{5}\right\rceil,\ 1 \leq i \leq r-1,\\
f(v_ix_{i+1}) &= \left\lceil\tfrac{4r+2}{5}\right\rceil,\ 1 \leq i \leq r-1,\\
f(u_{r+1}x_1) &= \left\lceil\tfrac{4r+2}{5}\right\rceil,\\
f(u_{r+2}x_r) &= \left\lceil\tfrac{4r+2}{5}\right\rceil,\\
f(x_iy_i) &= i,\ 1 \leq i \leq r.
\end{aligned}$$

The weight of vertices are:

$$\begin{aligned}
wt_f(y_i) &= i+1,\ 1 \leq i \leq r,\\
wt_f(u_i) &= \left\lceil\tfrac{4r+2}{5}\right\rceil + (i+1),\ 1 \leq i \leq r,\\
wt_f(u_{r+1}) &= 2\left\lceil\tfrac{4r+2}{5}\right\rceil + 2,\\
wt_f(u_{r+2}) &= 2\left\lceil\tfrac{4r+2}{5}\right\rceil + 3,\\
wt_f(v_i) &= 3\left\lceil\tfrac{4r+2}{5}\right\rceil + (i+3),\ 1 \leq i \leq r-1,\\
wt_f(x_i) &= 2\left\lceil\tfrac{4r+2}{5}\right\rceil + (i+3),\ 1 \leq i \leq r.
\end{aligned}$$

In Case 1, the weight of vertices are all distinct. The minimum weight is 2 and the maximum weight is $3\left\lceil\frac{4r+2}{5}\right\rceil + 3 + (r-1)$. We get upper bound $tvs(T_r(4,1)) \leq \left\lceil\frac{4r+2}{5}\right\rceil$. Combining with the lower bound (2) and the upper bound, we obtain $tvs(T_r(4,1)) = \left\lceil\frac{4r+2}{5}\right\rceil$.

Case 2. For $r \equiv 1 \mod \left\lceil \frac{4r+2}{5} \right\rceil$ $(7 \leq r \leq 11)$.

In Case 2, we construct label of vertices in the following way:

$$\begin{aligned}
f(u_i) &= 1,\ 1 \leq i \leq r-2,\\
f(u_{r-1}) &= 2,\\
f(u_r) &= 3,\\
f(u_{r+1}) &= 4,\\
f(u_{r+2}) &= 5,\\
f(v_i) &= 5,\ 1 \leq i \leq r-2,\\
f(v_{r-1}) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\\
f(x_i) &= 3,\ 1 \leq i \leq r-1,\\
f(x_r) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\\
f(y_i) &= 1,\ 1 \leq i \leq r-1,\\
f(y_r) &= 2,
\end{aligned}$$

Further, label of edges are defined as follows:

$$\begin{aligned}
f(u_1u_{r+1}) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\\
f(u_ru_{r+1}) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\\
f(u_iv_i) &= i+1,\ 1 \leq i \leq r-2,\\
f(u_{r-1}v_{r-1}) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\\
f(u_{i+1}v_i) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\ 1 \leq i \leq r-1,\\
f(v_ix_i) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\ i \leq r-1,\\
f(v_ix_{i+1}) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\ 1 \leq i \leq r-1,\\
f(u_{r+1}x_1) = f(u_{r+2}x_r) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\\
f(x_iy_i) &= i,\ 1 \leq i \leq r-1,\\
f(x_ry_r) &= \left\lceil \tfrac{4r+2}{5} \right\rceil.
\end{aligned}$$

Under the labeling f, we get:

$$\begin{aligned}
wt_f(y_i) &= i+1,\ 1 \leq i \leq r-1,\\
wt_f(y_r) &= 2 + \left\lceil \tfrac{4r+2}{5} \right\rceil,\\
wt_f(u_i) &= \left\lceil \tfrac{4r+2}{5} \right\rceil + (i+1),\ 1 \leq i \leq r-2\\
wt_f(u_{r-1}) &= 2 + \left\lceil \tfrac{4r+2}{5} \right\rceil,\\
wt_f(u_r) &= 3 + \left\lceil \tfrac{4r+2}{5} \right\rceil,\\
wt_f(u_{r+1}) &= 2\left\lceil \tfrac{4r+2}{5} \right\rceil + 4,\\
wt_f(u_{r+2}) &= 2\left\lceil \tfrac{4r+2}{5} \right\rceil + 5,\\
wt_f(v_i) &= 3\left\lceil \tfrac{4r+2}{5} \right\rceil + (i+6),\ 1 \leq i \leq r-2,\\
wt_f(v_{r-1}) &= 5\left\lceil \tfrac{4r+2}{5} \right\rceil,\\
wt_f(x_i) &= 2\left\lceil \tfrac{4r+2}{5} \right\rceil + (i+5),\ 1 \leq i \leq r-1,\\
wt_f(x_r) &= 4\left\lceil \tfrac{4r+2}{5} \right\rceil.
\end{aligned}$$

In Case 2, it is obvious that the weight of vertices are different. Thus, $tvs(T_r(4,1)) \leq \left\lceil \frac{4r+2}{5} \right\rceil$. Combining with (2), we get $tvs(T_r(4,1)) = \left\lceil \frac{4r+2}{5} \right\rceil$.

Case 3. For $r \equiv s \mod \left\lceil \frac{4r+2}{5} \right\rceil$ with $s \geq 2$.

We construct label of vertices as follows:

$$
\begin{aligned}
f(u_i) &= 1,\ 1 \leq i \leq r-2s,\\
f(u_j) &= f(u_{j-1}) + 1,\ r-2s+1 \leq j \leq r-1,\\
f(u_r) &= 2s+1,\\
f(u_{r+1}) &= 2s+2,\\
f(u_{r+2}) &= 2s+3,\\
f(v_i) &= 2s+3,\ 1 \leq i \leq r-2s,\\
f(v_{r-2s+1}) &= 2s+4,\\
f(v_{r-2s+2}) &= 2s+5,\ldots,\\
f(v_{r-1}) &= 2s + f(u_{r+1}),\\
f(v_r) &= 2s+3,\\
f(x_i) &= 2s+3,\ 1 \leq i \leq r-s,\\
f(x_{r-s+1}) &= 2s+4,\\
f(x_{r-s+2}) &= 2s+5,\ldots,\\
f(x_r) &= 3s+3,\\
f(y_i) &= 1,\ 1 \leq i \leq r-s,\\
f(y_{r-s+1}) &= 2,\\
f(y_{r-s+2}) &= 3,\ldots,\\
f(y_r) &= s+1,
\end{aligned}
$$

Meanwhile, label of edges are:

$$
\begin{aligned}
f(u_1u_{r+1}) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\\
f(u_ru_{r+1}) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\\
f(u_iv_i) &= i+s,\ 1 \leq i \leq r-2s,\\
f(u_iv_i) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\ r-2s+1 \leq i \leq r-1,\\
f(u_{i+1}v_i) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\ 1 \leq i \leq r-1,\\
f(x_iv_i) = f(x_{i+1}v_i) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\ 1 \leq i \leq r-1,\\
f(u_{r+1}x_1) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\\
f(u_{r+2}x_r) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\\
f(x_iy_i) &= i,\ 1 \leq i \leq r-s,\\
f(x_iy_i) &= \left\lceil \tfrac{4r+2}{5} \right\rceil,\ r-s+1 \leq i \leq r.
\end{aligned}
$$

In Case 3, we obtain the weight of edges:

$$wt_f(y_i) = i + 1, 1 \le i \le r - s;$$
$$wt_f(y_{r-s+i}) = r - s + i + 1, 1 \le i \le s.$$
$$wt_f(u_i) = \left\lceil \frac{4r+2}{5} \right\rceil + (i + s + 1), 1 \le i \le r - 2s;$$
$$wt_f(u_{2-2s+i}) = 2\left\lceil \frac{4r+2}{5} \right\rceil + (i + 1), 1 \le i \le 2s;$$
$$wt_f(u_{r+1}) = 2\left\lceil \frac{4r+2}{5} \right\rceil + (2s + 2);$$
$$wt_f(u_{r+2}) = 2\left\lceil \frac{4r+2}{5} \right\rceil + (2s + 3).$$
$$wt_f(v_i) = 3\left\lceil \frac{4r+2}{5} \right\rceil + (3s + i + 3), 1 \le i \le r - 2s;$$
$$wt_f(v_{r-2s+i}) = 4\left\lceil \frac{4r+2}{5} \right\rceil + (3s + i), 1 \le i \le 2s - 1.$$
$$wt_f(x_i) = 2\left\lceil \frac{4r+2}{5} \right\rceil + (2s + i + 3), 1 \le i \le r - s;$$
$$wt_f(x_{r-s+i}) = 3\left\lceil \frac{4r+2}{5} \right\rceil + (3s + i), 1 \le i \le s.$$

It is also clear that we get different weights of vertices, i.e., $\{2, 3, \ldots, 4\left\lceil \frac{4r+2}{5} \right\rceil + (5s-1)\}$. This shows the upper bound $tvs(T_r(4,1)) \le \left\lceil \frac{4r+2}{5} \right\rceil$. According to (2), the exact value is $\text{tvs}(T_r(4,1)) = \left\lceil \frac{4r+2}{5} \right\rceil$ and the proof is complete.

2.2. Computational result

In this part, computational result of determination of tvs of tadpole chain graph $T_r(4,1)$ is proposed. We construct computer program by using Matlab R2016a with an algorithm displayed in Algorithm 1.

Algorithm 1: ALGORITHM TO DETERMINE TVS OF TADPOLE CACTUS CHAIN GRAPH $T_r(4,1)$.

Input: length of chain r

Output: Weights of $u_i; u_{r+1}; u_{r+2}; v_i; x_i; y_i$ and tvs of $T_r(4,1)$

for $i = 1$ ***to*** r **do**
  label $uu(i) = 1$
  label $x(i) = 3$;label $y(i) = 1$ (% Labeling vertices of $T_r(4,1)$)
end
Set label $u_r = 2$; label $u_{r+1} = 2$;Label u=[(label uu)';label u_r ;label u_{r+1}]
for $i = 1$ ***to*** $r-1$ **do**
  label $v(i) = 3$
end
Set $u_1 = 1; u_{r+1} = 2; e1 = [u_1 u_{r+1}]$; label $e1 = \text{ceil}((4*r+2)/5)$; $u_r = 1; u_{r+2} = 3; e2 = [u_r u_{r+2}]$; label $e2 = \text{ceil}((4*r+2)/5)$ (% Labeling edges of $T_(4,1)$)
for $i = 1$ ***to*** $r-1$ **do**
  $u_j(i) = 1; v_j(i) = 3; e3 = [u_j; v_j]$; label $e3 = i$
  $u_{j+1}(i) = 1; v_j(i) = 3; e4 = [u_{j+1}; v_j]$; label $e4 = \text{ceil}((4*r+2)/5)$
  $v_j(i) = 3; x_j(i) = 3; e5 = [v_j; x_j]$;label $e5 = \text{ceil}((4*r+2)/5)$; $x_{j+1}(i) = 3; e6 = [v_j; x_{j+1}]$; label $e6 = \text{ceil}((4*r+2)/5)$
end
Set $u_{r+1} = 2; x_1 = 3; e61 = [u_{r+1}; x_1]$; label $e61 = \text{ceil}((4*r+2)/5)$; $u_{r+2} = 3; x_r = 3; e62 = [u_{r+2}; x_r]$;label $e62 = \text{ceil}((4*r+2)/5)$
for $i = 1$ ***to*** r **do**
  $x_j(i) = 3; y_j(i) = 3; e7 = [x_j; y_j]$; label $e7(i) = i$
end
for $i = 1$ ***to*** r **do**
  $Wy(i) = i+1$; $Wx(i) = 2*ceil((4*r+2)/5)+i+3$;$Wu(i) = ceil((4*r+2)/5)+i+1$ (Determining weight of vertices of $T_(4,1)$)
end
Set $Wu_{r+1} = 2*ceil((4*r+2)/5)+2$; $Wu_{r+2} = 2*ceil((4*r+2)/5)+3$
for $i = 1$ ***to*** $r-1$ **do**
  $Wv(i) = 3*ceil((4*r+2)/5)+i+3$
end

As a simulation, labeling of $T_6(4,1)$ is illustrated in Fig.1. By using the algorithm, the output of computer program for labeling and tvs of $T_6(4,1)$ is given in Fig. 2.

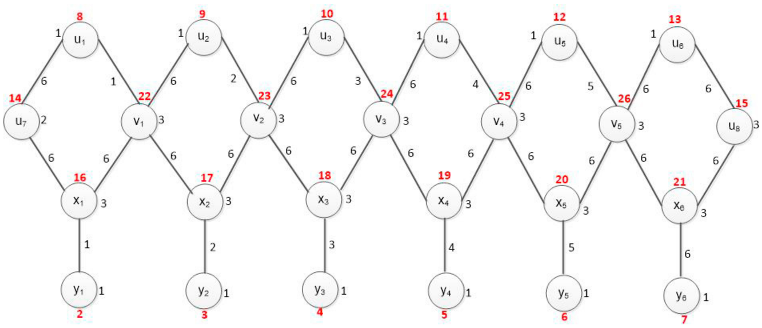

Fig. 1: Vertex irregular total 6-labeling of $T_6(4,1)$

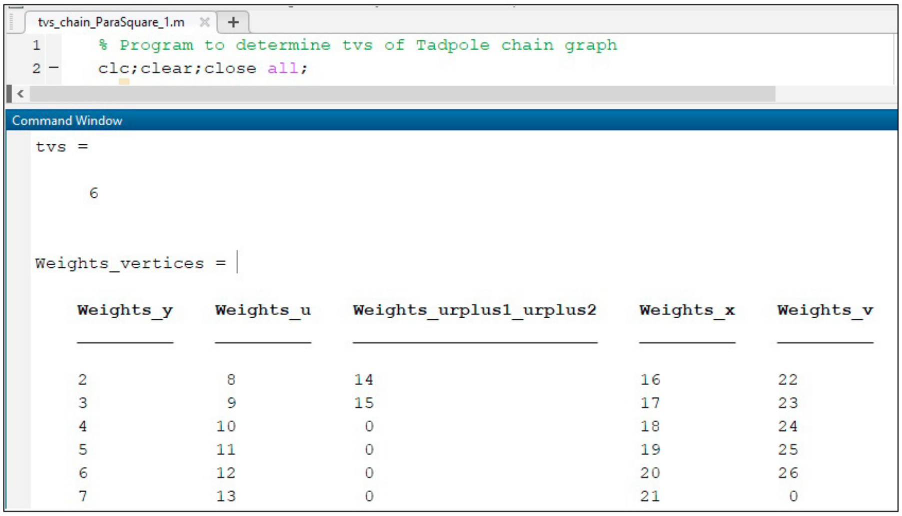

Fig. 2: Output of the algoritm in determining weight of vertices and tvs of tadpole chain graph $T_6(4,1)$

3. Conclusions

In this article, we have investigated tvs of tadpole chain graph and have proved $\text{tvs}(T_r(4,1)) = \left\lceil \frac{4r+2}{5} \right\rceil$. Moreover, we have verified the formulas of label of vertices, label of edges, weight of vertices, and exact value of tvs by using an algorithm. In upcoming work, we will determine tvs of generalized tadpole chain graph and related chain graphs. Also, we will construct a related algorithm.

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