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A BELIEF DEGREE CONSTRAINED PROGRAMMING MODEL FOR MAXIMUM CUT PROBLEM WITH UNCERTAIN EDGE WEIGHTS

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Abstract. *The traditional maximum cut problem assumes that the edge weights are crisp values or random variables. However, the edge weight in maximum cut problem has real and concrete implications in real life application, so it is more suitable to characterize the edge weights as uncertain variables. In this paper, the maximum cut problem in uncertain environment is considered. We originally propose a belief degree constrained programming model for uncertain maximum cut problem. Furthermore, we convert the model into its equivalent deterministic form which can be solved by classic programming methods. Finally, a numerical example is presented to show the application of the model.*
Keywords: Maximum cut problem, Belief degree constrained programming, Uncertainty theory, Uncertain variable

1. **Introduction.** The maximum cut problem is one of the most important and well-known combinatorial optimization problems. The maximum cut problem can be described as below. For an undirected edge-weighted graph with nonnegative weights, the goal of this problem is to find a partition of vertices into two disjoint sets such that the sum weight of the edges that join the two sets is as large as possible. This problem has been intensively studied not only for the theoretical interests but also due to its wide applicability in various fields such as statistical physics [1], layer assignment [2], and wireless sensor network [3]. For relevant literature and recent results about the maximum cut problem, the reader is referred to Poljak and Tuza [4] and references therein.

In early years, maximum cut problem has been investigated in a deterministic environment, in which the edge weights are crisp values. With being applied in various real-world fields, the weight of the edge in the maximum cut problem has real and concrete implications. In these cases, it is unsuitable to regard these indeterminacy factors as fixed quantities and to employ classical methods to study the maximum cut problem. Hence, much recent work introduced stochastic factors to maximum cut problem. For example, Poljak and Tuza [5] studied the expected relative error of a linear relaxation of the maximum cut problem in the random graph. After that, Feige and Schechtman [6] investigated the optimality of the random hyperplane rounding technique for maximum cut problem. Recently, Kardoš et al. [7] considered the maximum edge-cuts in cubic random graphs.

Most literature on maximum cut problem characterized the involving uncertainty as randomness. In real life, due to the lack of historical data, describing the uncertainty as randomness is not reasonable. In this case, we have to invite some domain experts to evaluate their belief degree about the edge weight. It has been demonstrated by Liu [8] that if we insist on dealing with the belief degree by using probability theory, some counterintuitive phenomena may happen. In order to deal with such indeterminacy factors, Liu [9] founded uncertainty theory, which has become a branch of axiomatic mathematics for dealing with human uncertainty.

As a useful tool to describe imprecise quantities in human systems, uncertainty theory has gained considerable achievement in practical aspect. Uncertain programming was pioneered by Liu [10] to deal with optimization problems with uncertain parameters. As an application, uncertain programming has been well developed and applied widely. For instance, Qin and Kar [11] presented a single-period inventory problem with uncertain demands. Chen et al. [12] discussed the minimum weight vertex covering problem in uncertain environment. Liu et al. [13] investigated the location problem of multi-product logistics distribution centers in uncertain environment. Lan et al. [14] studied the competitive logistics distribution center location problem instead of logistics distribution center location problem in uncertain environment. Yang et al. [15] established a furniture production planning model under uncertain environment for investigating how the loss averse customer's psychological satisfaction affects the company's furniture production planning. The interested readers can consult the book of Liu [16] for the comprehensive development of uncertainty theory.

The remaining sections are as follows. Section 2 describes some preliminary concepts of uncertainty theory for examining the present problem. Then in Section 3, we present the problem considered in this paper. In Section 4, a chance-constrained programming model for uncertain maximum cut problem is presented. Section 5 illustrates the proposed method by an example. Final conclusions and future researches are presented in Section 6.

2. **Preliminaries for Uncertainty Theory.** As this paper will investigate the maximum cut problem with uncertain edge weights under the framework of uncertainty theory, we briefly introduce some basic concepts and preliminary results in this field.

Definition 2.1. (Liu [9]) *Let L be a σ-algebra on a nonempty set Γ. A set function $M : L \to [0, 1]$ is called an uncertain measure if it satisfies the following three axioms*:
Axiom 1. ***(Normality Axiom)*** *$M\{\Gamma\} = 1$ for the universal set Γ*;
Axiom 2. ***(Duality Axiom)*** *$M\{\Lambda\} + M\{\Lambda^c\} = 1$ for any event Λ*;
Axiom 3. ***(Subadditivity Axiom)*** *For every countable sequence of events $\Lambda_1, \Lambda_2, \ldots$, we have*

$$M\left\{\bigcup_{i=1}^{\infty}\Lambda_i\right\} \le \sum_{i=1}^{\infty} M\{\Lambda_i\}.$$

The triplet (Γ, L, M) is called an uncertainty space. In order to obtain the uncertain measure of compound event, the product uncertain measure M on the product σ-algebra L was defined by Liu [17] as the following product axiom.
Axiom 4. ***(Product Axiom)*** *Let (Γ_k, L_k, M_k) be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure M is an uncertain measure satisfying*

$$M\left\{\prod_{k=1}^{\infty}\Lambda_k\right\} = \bigwedge_{k=1}^{\infty} M_k\{\Lambda_k\},$$

where Λ_k are arbitrarily chosen events from L_k for $k = 1, 2, \ldots$, respectively.

Definition 2.2. (Liu [9]) *An uncertain variable ξ is a measurable function from an uncertainty space (Γ, L, M) to the set of real numbers, i.e., for any Borel set B of real numbers, the set*

$$\{\xi \in B\} = \{\gamma \in \Gamma \mid \xi(\gamma) \in B\}$$

is an event.

The uncertainty distribution of an uncertain variable is defined by $\Phi(x) = M\{\xi \leq x\}$ for any real number x. The zigzag uncertain variable $\xi \sim Z(a, b, c)$ has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \\ (x-a)/2(b-a), & \text{if } a < x \leq b \\ (x+c-2b)/2(c-b), & \text{if } b < x \leq c \\ 1, & \text{if } x < c \end{cases}$$

where a, b, and c are real numbers with $a < b < c$.

Theorem 2.1. (Liu [18]) *Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If the function $f(x_1, x_2, \ldots, x_n)$ is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then*

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f\left(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)\right).$$

3. **Problem Formulation.** Given an edge-weighted undirected and simple graph $G = (V, E)$ with vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = \{e_{ij} = (v_i, v_j) \mid v_i \in V,\ v_j \in V\}$, each edge $e_{ij} \in E$ being associated with a weight ω_{ij}, and all the weights are presented by $\boldsymbol{\omega} = \{\omega_{ij} | e_{ij} \in E\}$. We say that the two nonempty subsets S and $\overline{S}$ partition a set V if $V = S \cup \overline{S}$, and $S \cap \overline{S} = \emptyset$. A partition of a set V into two subsets S and $\overline{S}$ is denoted by the unordered pair $\left(S, \overline{S}\right)$ (i.e., $\left(S, \overline{S}\right)$ and $\left(\overline{S}, S\right)$ represent the same partition). A partition of a graph $G(V, E)$ is a partition of its vertex set V. An edge $e_{ij} \in E$ is said to be cut by a partition $\left(S, \overline{S}\right)$ of G if its ends belong to two different subsets of the partition. A cut is denoted by $\delta(S) = \left\{e_{ij} \in E,\ v_i \in S,\ v_i \in \overline{S}\right\}$. Given a cut $\delta(S)$, the weight of the cut is the sum of the weights of the edges in the cut. Thus, the weight of a cut $\delta(S)$ is defined as

$$W(\delta(S)) = \sum_{e_{ij} \in \delta(S)} \omega_{ij}.$$

The maximum cut problem asks for the cut that maximizes the sum of the weights of its edges. It is well known that maximum cut problem can be formulated as follows:

$$\max_{S \subseteq V} \sum_{e_{ij} \in \delta(S)} \omega_{ij}. \tag{1}$$

By introducing cut vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ with $x_i = 1$ if $v_i \in S$ and $x_i = -1$ if $v_i \in \overline{S}$, Helmberg [19] stated that the problem (1) is equivalent to the following problem

$$\max_{\boldsymbol{x} \in \{-1,1\}^n} \sum_{i<j} \omega_{ij} \frac{1 - x_i x_j}{2}. \tag{2}$$

Then the weight of a cut $\delta(S)$ can be rewritten as

$$W(\delta(S)) = \frac{1}{4} \sum_{i=1}^{n} \sum_{j=1}^{n} \omega_{ij}(1 - x_i x_j).$$

More formally, the maximum cut problem can be written as the following quadratic programming problem,

$$\begin{cases} \max\limits_{\boldsymbol{x}} \ \dfrac{1}{4}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\omega_{ij}(1-x_ix_j) \\ \text{subject to:} \\ \qquad x_i \in \{-1,1\}, \ \ i=1,2,\ldots,n. \end{cases} \tag{3}$$

In the classic maximum cut problem, we always assume that the edge weights are crisp values. They are often not deterministic because decision makers are usually faced with some uncertain situations. If there is enough historical data of each edge weight, we can characterize the weight as a random variable and may create probability distributions of edge weight through statistic method. Unfortunately, sometimes we cannot obtain probability distributions of edge weights due to lack of historical data. In this situation, the edge weight data can be only obtained from the decision-makers' empirical estimation in a practical way. Therefore, it is inappropriate to regard subjective estimation weight data as random variables. Hence, in this paper, we assume that edge weights are all independent uncertain variables. That is, each edge weight w_{ij} is replaced by an uncertain variable ξ_{ij}, and all the weights can be presented by $\boldsymbol{\xi} = \{\xi_{ij}|e_{ij} \in E\}$.

Definition 3.1. *Let $G = (V, E)$ be an undirected and simple graph with uncertain edge weights. A cut $\delta^*(S)$ is called uncertain α-maximum cut if*

$$\max\left\{\overline{W} \mid M\left\{W(\delta^*(S)) \geq \overline{W}\right\} \geq \alpha\right\} \geq \max\left\{\overline{W} \mid M\left\{W(\delta(S)) \geq \overline{W}\right\} \geq \alpha\right\}$$

holds for any cut $\delta(S)$ of G, where $\alpha \in (0,1)$ is a predetermined confidence level.

4. **The Belief Degree Constrained Programming Model.** The philosophy of belief degree constrained programming, which was introduced by Peng et al. [20], is a powerful tool to deal with an indeterminacy system. Now we apply the belief degree constrained programming model to maximum cut problem in uncertain environment, and shall formulate an uncertain α-maximum cut model, which is shown as follows:

$$\begin{cases} \max\limits_{\boldsymbol{x}} \ \overline{W} \\ \text{subject to:} \\ \qquad M\left\{\dfrac{1}{4}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\xi_{ij}(1-x_ix_j) \geq \overline{W}\right\} \geq \alpha \\ \qquad x_i \in \{-1,1\}, \ \ i=1,2,\ldots,n. \end{cases} \tag{4}$$

Theorem 4.1. *Let $G = (V, E)$ be an undirected and simple graph with edge weights ξ_{ij} which are independent uncertain variables with regular uncertainty distributions Φ_{ij}, $i, j = 1, 2, \cdots, n$, respectively. Then the model (4) is equivalent to the following model*

$$\begin{cases} \max\limits_{\boldsymbol{x}} \ \dfrac{1}{4}\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}\Phi_{ij}^{-1}(1-\alpha)(1-x_ix_j) \\ \text{subject to:} \\ \qquad x_i \in \{-1,1\}, \ \ i=1,2,\ldots,n, \end{cases} \tag{5}$$

where Φ_{ij}^{-1} is the inverse uncertainty distributions of ξ_{ij}.

Proof: It is assumed that ξ_{ij} are independent uncertain variables with regular uncertainty distributions Φ_{ij}, $i, j = 1, 2, \ldots, n$, respectively. Then, using the inverse uncertainty distribution, we can transform the constraint

$$M\left\{\frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\xi_{ij}(1-x_ix_j)\geq\overline{W}\right\}\geq\alpha$$

into a deterministic constraint

$$\frac{1}{4}\sum_{i=1}^{n}\sum_{j=1}^{n}\Phi_{ij}^{-1}(1-\alpha)(1-x_ix_j)\geq\overline{W}.$$

Then we can easily prove that the model (4) can be equivalently transformed into the following deterministic model:

$$\begin{cases}\max\limits_{\boldsymbol{x}}\ \ \overline{W}\\ \text{subject to:}\\ \qquad\dfrac{1}{4}\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\Phi_{ij}^{-1}(1-\alpha)(1-x_ix_j)\geq\overline{W}\\ \qquad x_i\in\{-1,1\},\ \ i=1,2,\ldots,n.\end{cases}\tag{6}$$

Clearly, model (6) is equivalent to model (5). The theorem is proved.

5. **Numerical Example.** In this section, we consider a numerical example to illustrate the proposed uncertain α-maximum cut model. The example is shown in Figure 1; there are totally 8 vertices and 13 edges. Assume that all edge weights are zigzag uncertain variables ξ_{ij}. The distributions of ξ_{ij} are listed in Table 1.

When α=0.9, we can calculate $\Phi_{ij}^{-1}(0.1)$ for each ξ_{ij}. The values are listed in Table 2.

According to the model (4), the 0.9-maximum cut problem can be formulated as follows:

$$\begin{cases}\max\limits_{\boldsymbol{x}}\ \ \overline{W}\\ \text{subject to:}\\ \qquad M\left\{\dfrac{1}{4}\displaystyle\sum_{i=1}^{8}\sum_{j=1}^{8}\xi_{ij}(1-x_ix_j)\geq\overline{W}\right\}\geq 0.9\\ \qquad x_i\in\{-1,1\},\ \ i=1,2,\ldots,n.\end{cases}\tag{7}$$

It follows from Theorem 4.1 that the model (7) is equivalent to the deterministic quadratic programming model:

$$\begin{cases}\max\limits_{\boldsymbol{x}}\ \ \dfrac{1}{4}\displaystyle\sum_{i=1}^{8}\sum_{j=1}^{8}\Phi_{ij}^{-1}(0.1)(1-x_ix_j)\\ \text{subject to:}\\ \qquad x_i\in\{-1,1\},\ \ i=1,2,\ldots,n.\end{cases}\tag{8}$$

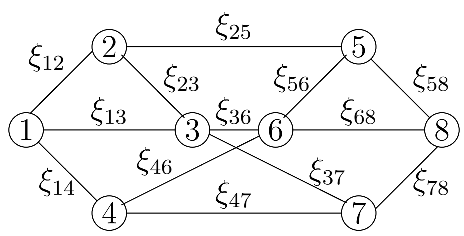

FIGURE 1. Uncertain weighted graph G for the numerical example (adopted from [21])

TABLE 1. The distributions of weights ξ_{ij} in Figure 1

ξ_{ij}	Φ_{ij}	ξ_{ij}	Φ_{ij}
ξ_{12}	$Z(2,3,4)$	ξ_{46}	$Z(5,7,8)$
ξ_{13}	$Z(3,4,5)$	ξ_{47}	$Z(3,5,7)$
ξ_{14}	$Z(2,3,5)$	ξ_{56}	$Z(2,4,5)$
ξ_{23}	$Z(2,5,7)$	ξ_{58}	$Z(4,6,7)$
ξ_{25}	$Z(5,6,7)$	ξ_{68}	$Z(7,8,9)$
ξ_{36}	$Z(5,7,9)$	ξ_{78}	$Z(2,4,6)$
ξ_{37}	$Z(2,3,6)$		

TABLE 2. List of $\Phi_{ij}^{-1}(0.1)$

ξ_{ij}	$\Phi_{ij}^{-1}(0.1)$	ξ_{ij}	$\Phi_{ij}^{-1}(0.1)$
ξ_{12}	2.2	ξ_{46}	5.4
ξ_{13}	3.2	ξ_{47}	3.4
ξ_{14}	2.2	ξ_{56}	2.4
ξ_{23}	2.6	ξ_{58}	4.4
ξ_{25}	5.2	ξ_{68}	7.2
ξ_{36}	5.4	ξ_{78}	2.4
ξ_{37}	2.2		

TABLE 3. List of α-maximum cut

α	$1-\alpha$	optimal solution x^*	maximum weight
0.9	0.1	$(1,-1,-1,-1,1,1,1,-1)^T$	43.2
0.8	0.2	$(1,-1,-1,-1,1,1,1,-1)^T$	46.4
0.7	0.3	$(1,-1,-1,-1,1,1,1,-1)^T$	49.6
0.6	0.4	$(1,-1,-1,-1,1,1,1,-1)^T$	52.8
0.5	0.5	$(1,-1,-1,-1,1,1,1,-1)^T$	56
0.4	0.6	$(1,1,-1,-1,-1,1,1,-1)^T$	59.6
0.3	0.7	$(1,1,-1,-1,-1,1,1,-1)^T$	63.2
0.2	0.8	$(1,1,-1,-1,-1,1,1,-1)^T$	66.8
0.1	0.9	$(1,1,-1,-1,-1,1,1,-1)^T$	70.4

The optimal solution of the model (8) can be obtained as $\boldsymbol{x}^* = (1,-1,-1,-1,1,1,1,-1)^T$ by using the values listed in Table 2 and LINGO solver. Then $S = \{v_1, v_5, v_6, v_7\}$, $\overline{S} = \{v_2, v_3, v_4, v_8\}$. So 0.9-maximum cut is $\{e_{12}, e_{13}, e_{14}, e_{25}, e_{36}, e_{37}, e_{46}, e_{47}, e_{58}, e_{68}, e_{78}\}$, and the maximum weight is 43.2. In order to investigate the influence of this parameter, the numerical example is further considered for different confidence levels. Choosing different α, we obtain Table 3. It can be seen from Table 3 that α has an effect on the optimal solutions, and the total weight of the maximum cut increases while the confidence level is decreasing.

6. **Conclusions.** This research proposed a belief degree programming model to handle the uncertain maximum cut problem, in which the edge weights were assumed to be uncertain variables. It has proved that there exists an equivalence relation between the uncertain maximum cut problem and the corresponding classic deterministic programming

problem under the framework of uncertainty theory. To test and verify the application of the proposed model, a numerical example was given in the end.

Our paper could also be used to incorporate other stylized facts and suggest several directions for future research. (1) It can investigate the maximum cut problem in other mixed indeterminate application environments, like uncertain random and random uncertain backgrounds. (2) This paper assumes that the vertex set is divided into two disjoint parts, but the dimensions of the two subsets are no extra constraint. Thus, it would be attractive to discuss the maximal bisection problem, where the dimensions of the two subsets are the same.

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