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A new approach for determining fuzzy chromatic number of fuzzy graph

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Abstract. A fuzzy graph referred in this paper is a graph with crisp vertex set and fuzzy edge set. The most important issue in the coloring problem of fuzzy graph is to construct a method for finding the chromatic number of fuzzy graph. Most of the methods that many researchers had been done still result crisp chromatic number. In this paper, we propose a new approach to determine fuzzy chromatic set of fuzzy graph. In our proposed method, the fuzzy chromatic set of fuzzy graph is constructed through its δ-chromatic number. Further, we investigate some properties of the fuzzy chromatic set of fuzzy graph. We show that fuzzy chromatic set of fuzzy graph is a discrete fuzzy number and then it is called by fuzzy chromatic number. To the best of our knowledge, no one has determined fuzzy chromatic number of fuzzy graph through its δ-chromatic number before now. Finally, a fuzzy chromatic algorithm based on the new approach is proposed.

Keywords: Fuzzy graph coloring, δ-chromatic number, fuzzy chromatic set, discrete fuzzy number, fuzzy chromatic number

1. Introduction

The vertex coloring problem of a graph is the problem of assigning a color to each vertex in such a way that the colors of its adjacent vertices are different and the number of colors used is minimized. The minimum number of colors used in the graph coloring is called a chromatic number. Some researchers focused on how to color a graph and how to obtain the chromatic number of a graph. The readers may refer to Ore [18], Werra and Hertz [5], Molloy and Reed [15] and Galinier et al. [6] for more detail in the methods for graph coloring. Further we call the classical graph by crisp graph.

The crisp graphs consist of vertices and edges which are all deterministic. In fact, the adjacency between two vertices is not completely deterministic because of the system may be much more complex in the real life situation. Therefore, the system cannot be handled by the classical graph and we need a tool to deal with these nondeterminacy phenomena.

Zadeh [8] handled the non-deterministic phenomena by the fuzzy set theory in 1965. After that, some researchers used the fuzzy set theory to handle the fuzzy phenomena in graphs. Kaufmann [1] proposed the concept of fuzzy graph in 1973. He introduced a fuzzy graph with crisp vertex set and fuzzy edge set. Further, Rosenfeld [3] developed the structures of fuzzy graphs in 1975 and he introduced another elaborated definition that is a fuzzy graph with fuzzy vertex set and fuzzy edge set. Recently, a lot of works on fuzzy graph had been done. Akram [11] initialized the concept of Bipolar fuzzy graphs in 2011. Next, he developed the structure of bipolar fuzzy graphs in 2013 [12]. Further, Akram et al. [13] and Akram et al. [14] gave some works on hypergraph in 2014.

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Some researchers introduced the methods for fuzzy graph coloring. Munoz et al.[19], Cioban [20], and Bershtein and Bozhenuk [10] gave the coloring methods for fuzzy graph with crisp vertex set and fuzzy edge set. In the year 2005, Munoz et al.[19] gave the method for fuzzy graph coloring in two different ways. In the first way, they proposed a method for fuzzy graph coloring by means of the α-cut of fuzzy graph. In the second way, they proposed the coloring function of fuzzy graph which depends on a distance defined between two colors. In the year 2007, Cioban [20] introduced the concept of fuzzy independent vertex set of fuzzy graph which depends on a value $\delta \in [0, 1]$. Next, he provided a method for fuzzy graph coloring based on the δ-fuzzy independent vertex set. An associated chromatic number was called δ-chromatic number which was a crisp number. Further, Bershtein and Bozhenuk [10] initialized a method for fuzzy graph coloring based on maximal fuzzy independent vertex set and introduced a fuzzy chromatic set of fuzzy graph. Most of the methods for fuzzy graph coloring, which had been done by some researchers, still result a crisp chromatic number.

In this paper, we deal with fuzzy graph with crisp vertex set and fuzzy edge set. A new approach to determine fuzzy chromatic set of fuzzy graph based on δ-chromatic number is proposed. Next, we show that the fuzzy chromatic set which is resulted by the new approach is equivalent to the fuzzy chromatic set as given in [10]. A property, which shows that the fuzzy chromatic set satisfies the characteristics of a discrete fuzzy number, is proved. Further, we call the fuzzy chromatic set by fuzzy chromatic number. To the best of our knowledge, till now no one has used δ-chromatic number to determine fuzzy chromatic number of fuzzy graph. Finally we propose a fuzzy chromatic algorithm which is developed from [1].

The rest of the paper is organized as follows. Section 2 presents some basic concepts in: graph theory, fuzzy set theory and fuzzy graph coloring. In Section 3, a new definition of fuzzy chromatic set of fuzzy graph is initialized. In Section 4, some properties of the fuzzy chromatic set of fuzzy graph are given. In the next section, we call it by fuzzy chromatic number. In Section 5, a fuzzy chromatic algorithm based on δ-chromatic number of fuzzy graph is proposed. An example is also given to illustrate the fuzzy chromatic algorithm. In Section 6, the application of fuzzy chromatic number of fuzzy graph is presented. Finally, the conclusions are given in Section 7.

2. Preliminaries

Some basic definitions in graph theory, fuzzy set theory and fuzzy graph coloring are reviewed in this section.

2.1. Basic concepts in graph theory

In this paper, every graph is assumed to be simple, finite and undirected graph. Let $G(V, E)$ be a graph with a nonempty vertex set $V = V(G)$ and an edge set $E = E(G)$. Each edge in G is an unordered pair of vertices of G. The number of vertices in the graph G is called the order of G. A subset of vertices $I \subseteq V$ is called an independent vertex set if $(u, v) \notin E$ for all $u, v \in I$. There are two different ways to define a graph coloring as given in [17]. In the first way, coloring of graph G is defined as a mapping $C : V \to N$ such that $C(u) \neq C(v)$ if $(u, v) \in E$. In the second way, coloring of graph G is defined as a partition of V into independent vertex sets $V_1, V_2, \ldots, V_k$, such that the subsets V_i are nonempty, $V_i \cap V_j = \emptyset$ for $i \neq j$ and $\bigcup_{i=1}^{k} V_i = V$. The minimum number of colors used in G is called the chromatic number of G.

Next, the algorithm introduced by Kishore and Sunitha [2] for finding the chromatic number of crisp graph is presented in this subsection. Later, this algorithm will be developed in order to find fuzzy chromatic number of fuzzy graph.

Let $G = (V, E)$ be a crisp graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and the edge set E. Let I be an independent vertex set in G.

Step 1: Initialize $k' = 0$.
Step 2: If $|V| = 0$, return $\chi(G) = 0$.
Step 3: Determine a family of independent vertex sets $\Gamma = \{I_1, I_2, \ldots, I_t\}$ where $I_j(u_i) = \{u_j\} \cup \{u_i | (u_i, u_j) \notin E, i \neq j\}$
Step 4: Choose a subfamily $\Gamma' = \{I_1, I_2, \ldots, I_k\}$ such that $I_i \cap I_j = \emptyset, i \neq j, i, j = 1, \ldots, k$ and $\Sigma_{j=1}^{k} |I_j|$ is maximum. Let $S = V - \bigcup_{j=1}^{k} I_j$.
Step 5: If $|S| = 0$, then:
If $k = 1$, then $\chi(G) = 1$ and go to Step 7.
Else if $1 < k \leq n$ then $\chi(G) = k$ and go to Step 7.
Step 6: If $|S| \neq 0$, then $\chi(G) = k$.
Let $k' = \chi(G) + k'$.
Put $G = \langle S \rangle$ and go to Step 3.

Step 7: $\chi(G) = k + k'$.
Step 8: End

2.2. Basic concepts in fuzzy set

We first review some concepts as given in [4], [7] and [8]. The notion of fuzzy set was introduced by Zadeh [8] in 1965. Let X be a non empty set of objects. A fuzzy set $\tilde{A}$ in X is a set of the form $\{(x, \mu_{\tilde{A}}(x)) : x \in X\}$, where $\mu_{\tilde{A}} : X \to [0, 1]$ is a membership function of the fuzzy set $\tilde{A}$. The support of $\tilde{A}$, denoted by $S(\tilde{A})$, is the crisp set given by $S(\tilde{A}) = \{x \in X | \mu_{\tilde{A}}(x) > 0\}$. The height of $\tilde{A}$, denoted by $h(\tilde{A})$, is defined as $h(\tilde{A}) = \sup\{\mu_{\tilde{A}}(x)|x \in X\}$. If $h(\tilde{A}) = 1$ then the fuzzy set $\tilde{A}$ is called a normal fuzzy set, otherwise it is called a sub normal one. Let $\alpha \in (0, 1]$, the α-cut of the fuzzy set $\tilde{A}$ is a crisp set $\tilde{A}_\alpha = \{x \in X|\mu_{\tilde{A}}(x) \geq \alpha\}$. Meanwhile, $\tilde{A}_0 = \overline{\{x \in X|\mu_{\tilde{A}}(x) > 0\}}$ is said to be a closure of support $\tilde{A}$. Let $\tilde{A}$ be fuzzy set on X with membership function $\mu_{\tilde{A}} : X \to [0, 1]$ and $\tilde{B}$ be fuzzy set on X with membership function $\mu_{\tilde{B}} : X \to [0, 1]$. The fuzzy set $\tilde{A}$ is called subset of $\tilde{B}$, denoted by $\tilde{A} \subseteq \tilde{B}$, if $\mu_{\tilde{A}}(x) \leq \mu_{\tilde{B}}(x)$ for all $x \in X$. The union of $\tilde{A}$ and $\tilde{B}$, denoted by $\tilde{A} \cup \tilde{B}$, is the fuzzy set $\tilde{C}$ on X with the membership function $\mu_{\tilde{C}} : X \to [0, 1]$ defined by $\mu_{\tilde{C}}(x) = \max\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$ for all $x \in X$. The intersection of $\tilde{A}$ and $\tilde{B}$, denoted by $\tilde{A} \cap \tilde{B}$, is the fuzzy set $\tilde{D}$ on X with the membership function $\mu_{\tilde{D}} : X \to [0, 1]$ defined by $\mu_{\tilde{D}}(x) = \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{B}}(x)\}$ for all $x \in X$.

The concepts of fuzzy number and discrete fuzzy number are given in Definition 1 and Definition 2, respectively.

Definition 2.1. [4]. A fuzzy set $\tilde{A}$, defined on the set of real numbers R, is said to be a fuzzy number if it satisfies the following conditions:

i $\tilde{A}$ is normal, i.e.$\exists x_0 \in R$ such that $\mu_{\tilde{A}}(x_0) = 1$.
ii $\tilde{A}_\alpha$ is a closed interval for every $\alpha \in (0, 1]$.
iii The support of $\tilde{A}$ is bounded.

Definition 2.2. [7]. Let $C \subseteq R$ be a countable set. A fuzzy set $\tilde{A}$ is called discrete fuzzy number on C if it satisfies the following conditions:

i the set $\tilde{A}_0 \subset C$ and it is finite;
ii there exists $x_0 \in C$ such that $\mu_{\tilde{A}}(x_0) = 1$;
iii $\mu_{\tilde{A}}(x_s) \leq \mu_{\tilde{A}}(x_t)$ for any $x_s, x_t \in C$ with $x_s \leq x_t \leq x_0$;
iv $\mu_{\tilde{A}}(x_s) \geq \mu_{\tilde{A}}(x_t)$ for any $x_s, x_t \in C$ with $x_0 \leq x_s \leq x_t$.

2.3. Fuzzy graph coloring

In this section, some basic concepts in fuzzy graph, as given in [16], are reviewed. Further, some concepts in fuzzy graph coloring, as given in [9], [10], and [20] are presented.

Let V be a finite nonempty set and $E \subseteq V \times V$. A fuzzy graph $\tilde{G}(V, \tilde{E})$ is a graph having a crisp vertex set V and a fuzzy edge set $\tilde{E}$ with a membership function $\mu : V \times V \to [0, 1]$. Meanwhile, a fuzzy graph $\tilde{G}(\tilde{V}, \tilde{E})$, is a graph having a fuzzy vertex set $\tilde{V}$ with a membership function $\sigma : V \to [0, 1]$ and a fuzzy edge set $\tilde{E}$ with a membership function $\mu : V \times V \to [0, 1]$ such that $\mu(u, v) \leq min\{\sigma(u), \sigma(v)\}$ for all $u, v \in V$. In this paper, we deal with a fuzzy graph $\tilde{G}(V, \tilde{E})$. Further, a graph $G(V, E)$ will be called a crisp graph.

Let $\tilde{G}'(V', \tilde{E}')$ be fuzzy graph having crisp vertex set V' and fuzzy edge set $\tilde{E}'$ with membership function $\mu_{\tilde{E}'} : V' \times V' \to [0, 1]$. The fuzzy graph $\tilde{G}'$ is called fuzzy subgraph of $\tilde{G}$ if $V' \subseteq V$ and $\tilde{E}' \subseteq \tilde{E}$. Let $\tilde{H}(P, \tilde{E}_H)$ be fuzzy graph having crisp vertex set P and fuzzy edge set $\tilde{E}_H$ with membership function $\mu_{\tilde{E}_H} : P \times P \to [0, 1]$. The fuzzy graph $\tilde{H}$ is called a fuzzy subgraph of $\tilde{G}$ induced by P, denoted by $\langle P \rangle$, if $P \subseteq V$ and $\mu_{\tilde{E}_H}(u, v) = \mu(u, v)$ for $u, v \in P$.
Meanwhile, the underlying crisp graph of fuzzy graph $\tilde{G}(V, \tilde{E})$ is the graph $G^*(V^*, E^*)$ where $V^* = V$ and $E^* = \{(u, v) \in V \times V : \mu(u, v) > 0\}$.

Firstly, the method for finding fuzzy chromatic set of fuzzy graph was proposed by Bershtein and Bozhenuk [10]. In order to define the fuzzy chromatic set, they introduced the concepts of fuzzy independent vertex set, maximal fuzzy independent vertex set and separation degree of fuzzy graph.

Definition 2.3. [9]. Let $\tilde{G}(V, \tilde{E})$ be a fuzzy graph. A subset of vertices $X \subseteq V$ is said to be a fuzzy independent vertex set (fuzzy internal stable set) with a degree of independence $\alpha(X) = 1 - \max\{\mu(x, y)|x, y \in X\}$.

Definition 2.4. [9]. A subset $X \subseteq V$ of fuzzy graph $\tilde{G}(V, \tilde{E})$ is said to be a maximal fuzzy independent vertex set with the degree of independence $\alpha(X)$, if $\alpha(X') < \alpha(X)$ for any fuzzy independent vertex set $X' \supset X$.

Next, in order to define fuzzy chromatic set of fuzzy graph, a concept of separation degree of fuzzy graph was given in [10].

Definition 2.5. [10]. Let $\tilde{G}(V, \tilde{E})$ be a fuzzy graph with n vertices. Let $\mathcal{A}$ be a family of maximal fuzzy independent vertex sets of $\tilde{G}$. Let $\mathcal{F} = \{\tilde{A}_{\tilde{G}}\}$ be a family which consist of fuzzy sets $\tilde{A}_{\tilde{G}}$, where $\tilde{A}_{\tilde{G}} = \{(k, L_{\tilde{A}}(k))|k = 1, ..., n\}$. The value

$$L_{\tilde{A}}(k) = \min\{\alpha(X_1), \alpha(X_2), \ldots, \alpha(X_k)|X_i \in \mathcal{A}\}$$

is called a separation degree of fuzzy graph $\tilde{G}$ with k colors.

Definition 2.6. [10]. A fuzzy set $\tilde{\chi}_{\tilde{G}} \in \mathcal{F}$, where $\tilde{\chi}_{\tilde{G}} = \{(k, L_{\tilde{\chi}}(k))|k = 1, ..., n\}$ and $L_{\tilde{\chi}}(k) = \min\{\alpha(X'_1), \alpha(X'_2), \ldots, \alpha(X'_k)|X'_i \in \mathcal{A}\}$, is called fuzzy chromatic set of fuzzy graph $\tilde{G}$ if

$$\tilde{A}_{\tilde{G}} \subseteq \tilde{\chi}_{\tilde{G}}$$

for every $\tilde{A}_{\tilde{G}} \in \mathcal{F}$. In other words, $(\forall \tilde{A}_{\tilde{G}} \in \mathcal{F})(\forall k = 1, ..., n)(L_{\tilde{A}}(k) \leq L_{\tilde{\chi}}(k))$.

Furthermore, Bershtein and Bozhenuk [10] gave a proposition to determine fuzzy chromatic set of fuzzy graph as follows.

Proposition 2.7. [10]. Let $\tilde{G}(V, \tilde{E})$ be a fuzzy graph with n vertices. A fuzzy set

$$\tilde{\chi}_{\tilde{G}} = \left\{(k, L_{\tilde{\chi}}(k))|k = 1, \ldots, n\right\},$$

is a fuzzy chromatic set of $\tilde{G}$ if and only if there is no more than k'-maximal fuzzy independent vertex sets $X_1, X_2, \ldots, X_{k'}$ ($k' \leq k$), with degree of independence $\alpha_1 = \alpha(X_1), \alpha_2 = \alpha(X_2), \ldots, \alpha_{k'} = \alpha(X_{k'})$ such that:

1) $\min\{\alpha_1, \alpha_2, \ldots, \alpha_{k'}\} = L_{\tilde{\chi}}(k)$;
2) $\cup_{j=1}^{k'} X_j = V$;
3) there is no other family $\{X'_1, X'_2, \ldots, X'_{k''}\}$ with $k'' \leq k$ for which $\min\{\alpha'_1, \alpha'_2, \ldots, \alpha'_{k''}\} > \min\{\alpha_1, \alpha_2, \ldots, \alpha_{k'}\}$ and the property 2) is satisfied.

Based on Proposition 2.7, there were two steps for finding fuzzy chromatic set of fuzzy graph $\tilde{G}(V, \tilde{E})$ [10]. The first step was to determine the family of maximal fuzzyindependent vertex sets of $\tilde{G}$. The second step was to select k-maximal fuzzy independent vertex sets which gave the maximum value of the separation degree. It is clear that the two steps are not simple steps and these are time consuming works.

In this paper, we give a new approach to determine fuzzy chromatic set of fuzzy graph based on δ-chromatic number which is a crisp number. By using the new approach, the step for determining fuzzy chromatic number of fuzzy graph becomes simpler and it is not a time consuming work.

The concept of δ-chromatic number was given by Cioban [20]. In order to determine the δ-chromatic number of fuzzy graph, the concept of δ-fuzzy independent vertex set was introduced [20].

Definition 2.8. [20] Let $\tilde{G}(V, \tilde{E})$ be a fuzzy graph and $\delta \in [0, 1]$. A fuzzy independent vertex set S is a set where $\mu(u, v) \leq \delta$ for all $u, v \in S$. The fuzzy independent vertex set S can be called by δ-fuzzy independent vertex set and it is denoted by S^{δ}.

For $\delta = 0$, the δ-fuzzy independent vertex set S^0 is a set where $\mu(u, v) = 0$ for all $u, v \in S^0$. If the fuzzy graph $\tilde{G}(V, \tilde{E})$ becomes the underlying crisp graph $G^*(V^*, E^*)$, then the δ-fuzzy independent vertex set S^0 is a set where each pair of vertices is not adjacent. In other words, the fuzzy independent vertex set S^0 becomes an independent vertex set of the underlying crisp graph G^*.

Next, the concepts of k-coloring and δ-chromatic number of fuzzy graph as given in [20] are presented as follows.

Definition 2.9. [20] Let $\delta \in [0, 1]$. The k-coloring of fuzzy graph $\tilde{G}(V, \tilde{E})$ is defined as a partition of V into $k-$ fuzzy independent vertex sets $\{S_1^{\delta}, ..., S_k^{\delta}\}$ such that $S_i^{\delta} \cap S_j^{\delta} = \emptyset$ for all $i \neq j$ and $S_1^{\delta} \cup ... \cup S_k^{\delta} = V$. An associated chromatic number which is called δ-chromatic number, denoted by $\chi^{\delta}(\tilde{G})$, is defined as the smallest natural number k used in the k-coloring of fuzzy graph $\tilde{G}$.

If $\delta = 0$, then each δ-fuzzy independent vertex sets S_i^0 becomes crisp independent vertex set S_i. Since then, the partition $\{S_1^0, ..., S_k^0\}$ becomes a partition $\{S_1, S_2, ..., S_k\}$ of the underlying crisp graph G^*. Thus, the δ-chromatic number $\chi^0(\tilde{G})$ becomes a chromatic number $\chi(G^*)$ of the underlying crisp graph G^*.

At the end of this section, we summary the list of used notations in Table 1.

3. New definition of fuzzy chromatic set of fuzzy graph

A new approach to determine fuzzy chromatic set of fuzzy graph is presented in Definition 10. The new approach is motivated by the concept of δ chromatic number of fuzzy graph as given in [20].

Definition 3.1. Let $\tilde{G}(V, \tilde{E})$ be a fuzzy graph with n vertices. A fuzzy set

$$\tilde{\chi}(\tilde{G}) = \{(k, L_{\tilde{\chi}}(k))|k = 1, \ldots, n\}$$

Table 1
List of used notations in this paper

$\tilde{A}$	Fuzzy set
$\mu_{\tilde{A}}$	Membership function of fuzzy set $\tilde{A}$
$h(\tilde{A})$	Height of fuzzy set $\tilde{A}$
$S(\tilde{A})$	Support of fuzzy set $\tilde{A}$
$\tilde{A} \cup \tilde{B}$	Union of fuzzy set $\tilde{A}$ and $\tilde{B}$
$\tilde{A} \cap \tilde{B}$	Intersection of fuzzy set $\tilde{A}$ and $\tilde{B}$
$\tilde{A} \subseteq \tilde{B}$	Fuzzy set $\tilde{A}$ is subset of fuzzy set $\tilde{B}$
$G(V, E)$	Crisp graph G
$\tilde{G}(V, \tilde{E})$	Fuzzy graph with crisp vertex set V and fuzzy edge set $\tilde{E}$
$\tilde{G}(\tilde{V}, \tilde{E})$	Fuzzy graph with fuzzy vertex set V and fuzzy edge set $\tilde{E}$
$G^*(V^*, E^*)$	Underlying crisp graph of fuzzy graph $\tilde{G}$
$\langle P \rangle$	Fuzzy subgraph of $\tilde{G}(V, \tilde{E})$ induced by the vertex set $P \subseteq V$
$\alpha(X)$	Degree of independence of fuzzy independent vertex set X
$\mathcal{F}$	Family of fuzzy sets $\mathcal{F} = \{\tilde{A}_{\tilde{G}}\}$
$\tilde{A}_{\tilde{G}}$	Fuzzy set $\{(k, L_{\tilde{A}}(k))\|k = 1, ..., n\}$ of fuzzy graph $\tilde{G}$ with n vertices
$L_{\tilde{A}}(k)$	Separation degree of fuzzy graph $\tilde{G}$ with k colors, with respect to fuzzy set $\tilde{A}$
$\tilde{\chi}_{\tilde{G}}$	Fuzzy chromatic set of fuzzy graph $\tilde{G}$
$L_{\tilde{\chi}}(k)$	Separation degree of fuzzy graph G with k colors, with respect to fuzzy set $\tilde{\chi}$
$\tilde{\chi}(\tilde{G})$	Fuzzy chromatic set of fuzzy graph $\tilde{G}$ (the new definition)
S^δ	δ-fuzzy independent vertex set
S^0	δ-fuzzy independent vertex set for $\delta = 0$

is said to be a fuzzy chromatic set of $\tilde{G}$, denoted by $\tilde{\chi}(\tilde{G})$, if

$$L_{\tilde{\chi}}(k) = \max\{1 - \delta | \delta \in [0, 1], \chi^\delta(G) = k\}.$$

The value $L_{\tilde{\chi}}(k)$ represents the grade of membership of the number k in the fuzzy chromatic set $\tilde{\chi}$.

There are some remarks with respect to Definition 3.1.

Remark 3.1.1. The δ-chromatic number $\chi^0(\tilde{G}) = k_0$ with the degree $L_{\tilde{\chi}}(k_0) = 1$ in the fuzzy chromatic set $\tilde{\chi}(\tilde{G})$, is equal to the chromatic number $\chi(G^*)$ of the underlying crisp graph G^*.

Remark 3.1.2. Let $\tilde{G}(V, \tilde{E})$ be a fuzzy graph with n vertices and $V = \{v_1, ..., v_n\}$. The possible partition which gives $\chi^\delta(\tilde{G}) = n$ for all $\delta \in [0, 1]$ is only the set $\{S_1^\delta, \ldots, S_n^\delta\}$, where $S_i^\delta = \{v_i\}$ for all $i = 1, ..., n$. Thus, $L_{\tilde{\chi}}(n) = \max\{1 - \delta | \chi^\delta(\tilde{G}) = n\} = 1$.

Further, we show that the fuzzy chromatic set resulted by the new approach is equivalent to the fuzzy chromatic set as given in Definition 2.6.

Theorem 3.2. *Let $\tilde{G}(V, \tilde{E})$ be a fuzzy graph with n vertices. A fuzzy set*

$$\tilde{\chi}(\tilde{G}) = \{(k, L_{\tilde{\chi}}(k))k = 1, \ldots, n\},$$

where $L_{\tilde{\chi}}(k) = \max\{1 - \delta | \delta \in [0, 1], \chi^\delta(G) = k\}$, is a fuzzy chromatic set of $\tilde{G}$ if and only if there are k-maximal fuzzy independent vertex sets $X_1, X_2, \ldots, X_k$ with the degrees of independence $\alpha_1 = \alpha(X_1), \alpha_2 = \alpha(X_2), \ldots, \alpha_k = \alpha(X_k)$ respectively, such that:

1) $\min\{\alpha_1, \alpha_2, \ldots, \alpha_k\} = L_{\tilde{\chi}}(k)$;
2) $\cup_{j=1}^{k} X_j = V$;
3) there is no other family $\{X'_1, X'_2, \ldots, X'_{k'}\}$ with $k' \leq k$ such that $\min\{\alpha'_1, \alpha'_2, \ldots, \alpha'_{k'}\} > \min\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ and property 2) is true.

proof. Let $\tilde{\chi}(\tilde{G}) = \{(k, L_{\tilde{\chi}}(k))\}$ be a fuzzy chromatic set of $\tilde{G}$, where $L_{\tilde{\chi}}(k) = \max\{1 - \delta | \delta \in [0, 1], \chi^\delta(G) = k\}$.
In other words, there exist $\delta_t \in [0, 1]$ and a partition $\{S_1^{\delta_t}, S_2^{\delta_t}, \ldots, S_k^{\delta_t}\}$ of V such that $\chi^{\delta_t}(\tilde{G}) = k$ and $L_{\tilde{\chi}}(k) = 1 - \delta_t$. In order to prove that there are k-maximal fuzzy independent vertex sets, we consider two cases.

Case 1: $k = n$**.** There exists $\delta_t = 0$, such that the partition $S = \{S_1^{\delta_t}, S_2^{\delta_t}, \ldots, S_n^{\delta_t}\}$ gives $\chi^{\delta_t}(\tilde{G}) = n$ where $S_i^{\delta_t} = \{v_i\}$ for all $i = 1, \ldots, n$ with the degree $L_{\tilde{\chi}}(n) = 1 - \delta_t = 1$. Since there is no other fuzzy independent vertex set $S_i'^{\delta_t}$ such that $S_i^{\delta_t} \subset S_i'^{\delta_t}$ and $\alpha(S_i'^{\delta_t}) > 1$ for $i = 1, \ldots, n$, we have k-maximal fuzzy independent vertex sets $X_1 = S_1^{\delta_t}, \ldots, X_k = S_k^{\delta_t}$ where $\alpha(X_i) = 1$ for all $i = 1, \ldots, k$ and $k = n$. Hence $L_{\tilde{\chi}}(k) = 1 - \delta_t = 1 = \min\{\alpha(X_i)\}$. It is clear that, $\cup_{j=1}^{k} X_j = V$. Obviously, there is no other family $\{X'_1, \ldots, X'_k\}$ with $\min\{\alpha'_1, \alpha'_2, \ldots, \alpha'_{k'}\} > \min\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ and property 2) is true. Thus, the k-maximal fuzzy independent vertex sets $X_1, \ldots, X_k$ satisfy properties 1) $-$ 3).

Case 2: $k < n$**.** There exists δ_t such that the partition $S = \{S_1^{\delta_t}, S_2^{\delta_t}, \ldots, S_k^{\delta_t}\}$ gives $\chi^{\delta_t}(\tilde{G}) = k$ and $L_{\tilde{\chi}}(k) = 1 - \delta_t$. Obviously, $\max\{\mu(u, v) | u, v \in S_i^{\delta_t}\} \leq \delta_t$. Hence, $\alpha(S_i^{\delta_t}) \geq 1 - \delta_t$ for $i = 1, \ldots, k$.

We will construct k-maximal fuzzy independent vertex sets $X_1, \ldots, X_k$.

Firstly, we prove that $S_1^{\delta_t}$ is a maximal fuzzy independent vertex set. Suppose that $S_1^{\delta_t}$ is not a maximal fuzzy independent vertex set. By Definition 2.4, there exists a set $S_j^{\delta_t}$ such that $S_1^{\delta_t} \subset S_j^{\delta_t}$ with $\alpha(S_1^{\delta_t}) \leq \alpha(S_j^{\delta_t})$. We put $S_1^{\delta_t}$ and $S_j^{\delta_t}$ in the right order such that $S_j^{\delta_t} = S_2^{\delta_t}$. Since $S_1^{\delta_t} \subset S_2^{\delta_t}$, we can construct a set $S_{1'}^{\delta_t} = S_1^{\delta_t} \cup S_2^{\delta_t}$. Consequently, we obtain a partition $\{S_{1'}^{\delta_t}, S_{3'}^{\delta_t}, \ldots, S_{k'}^{\delta_t}\}$ such that $\chi^{\delta_t}(\tilde{G}) = k' < k$. It is a contradiction. Thus, $S_1^{\delta_t}$ is a maximal fuzzy independent vertex set. We put $X_1 = S_1^{\delta_t}$, with $\alpha(X_1) \geq 1 - \delta_t$.

Secondly, we construct maximal fuzzy independent vertex sets X_j for $j = 2, \ldots, k$ as follows:

$X_2 = S_2^{\delta_t} \cup \{x \in X_1 | \max\{\mu(x, y)\} \leq \delta_t, y \in S_2^{\delta_t}\}$,

$X_3 = S_3^{\delta_t} \cup \{x \in X_1 \cup X_2 | \max\{\mu(x, y)\} \leq \delta_t, y \in S_3^{\delta_t}\}$,

$X_4 = S_4^{\delta_t} \cup \{x \in X_1 \cup X_2 \cup X_3 | \max\{\mu(x, y)\} \leq \delta_t, y \in S_4^{\delta_t}\}$,

...

$X_k = S_k^{\delta_t} \cup \{x \in X_1 \cup X_2 \cup X_3 \cup \ldots \cup X_{k-1} | \max\{\mu(x, y)\} \leq \delta_t, y \in S_k^{\delta_t}\}$.

Obviously, there does not exist a set X_i such that $X_i \subseteq X_j$ for $i \neq j, i, j = 1, 2, \ldots, k$. Since otherwise, we have a partition $\{S_{1'}^{\delta_t}, S_{2'}^{\delta_t}, \ldots, S_{k'}^{\delta_t}\}$ such that $\chi^{\delta_t}(\tilde{G}) = k' < k$ and it is a contradiction. Hence, all of the sets X_j are maximal fuzzy independent vertex sets in $\tilde{G}$ for all $j = 1, \ldots, k$ with the degrees of independence $\alpha_j \geq 1 - \delta_t$.

Finally, we prove that the maximal fuzzy independent vertex sets $X_1, \ldots, X_k$ satisfy properties $1) - 3)$.

1) Since $\max\{\mu(u, v) | u, v \in X_i\} \leq \delta_t, \alpha(X_i) \geq 1 - \delta_t$. Hence, $L_{\tilde{\chi}}(k) = 1 - \delta_t = \min \alpha(X_i)$.
2) Obviously, $\bigcup_{j=1}^{k} X_j = V$.
3) By Definition 2.6, the value $L_{\tilde{\chi}}(k) = 1 - \delta_t = \min \alpha(X_i)$ is the biggest separation degree in the fuzzy graph $\tilde{G}$. Therefore, there does not exist a family $\{X'_1, X'_2, \ldots, X'_{k'}\}$ such that $\min\{\alpha'_1, \alpha'_2, \ldots, \alpha'_{k'}\} > \min\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$, $k' \leq k$ and property 2) is true.

Conversely, there are k-maximal fuzzy independent vertex sets $X_1, X_2, \ldots, X_k$ with the degrees of independence $\alpha_1, \alpha_2, \ldots, \alpha_k$ respectively which satisfy properties $1) - 3)$. We will prove that that the fuzzy set $\tilde{\chi}(\tilde{G}) = \{(k, L(k))\}$ where $L_{\tilde{\chi}}(k) = \max\{1 - \delta | \delta \in [0, 1], \chi^{\delta}(G) = k\}$ is the fuzzy chromatic set of $\tilde{G}$. In other words, we will show that there exist $\delta_t \in [0, 1]$ and a partition $\{S_1^{\delta_t}, \ldots, S_k^{\delta_t}\}$ such that $\chi^{\delta_t}(\tilde{G}) = k$ and $L_{\tilde{\chi}}(k) = 1 - \delta_t$.

Let $X_t \in \{X_1, X_2, \ldots, X_k\}$ such that $\alpha(X_t) = \min\{\alpha(X_1), \ldots, \alpha(X_k)\}$. In other words, the value of $\max\{\mu(u, v) | u, v \in X_t\}$ is the greatest value among the values of $\max\{\mu(u, v) | u, v \in X_i\}, i \neq t$. Let $\delta_t = \max\{\mu(u, v) | u, v \in X_t\}$. We have $\mu(u, v) \leq \delta_t$ for all $u, v \in X_i, i = 1, \ldots, k$.

Further, we construct δ_t-fuzzy independent vertex sets $S_1^{\delta_t}, \ldots, S_k^{\delta_t}$ as follows:

$S_1^{\delta_t} = X_1$,

$S_2^{\delta_t} = X_2 - X_1$,

$S_3^{\delta_t} = X_3 - \{X_1 \cup X_2\}$,

$S_4^{\delta_t} = X_4 - \{X_1 \cup X_2 \cup X_3\}$,

...

$S_k^{\delta_t} = X_k - \{X_1 \cup X_2 \cup X_3 \cup \ldots \cup X_{k-1}\}$.

The fuzzy independent vertex sets $S_1^{\delta_t}, S_2^{\delta_t}, \ldots, S_k^{\delta_t}$ satisfy $\bigcup_{j=1}^{k} S_j^{\delta_t} = V$ and $S_i^{\delta_t} \cap S_j^{\delta_t} = \emptyset$ for all $i \neq j$. Thus, there exist $\delta_t \in [0, 1]$ and a partition $\{S_1^{\delta_t}, S_2^{\delta_t}, \ldots, S_k^{\delta_t}\}$ of V such that $\chi^{\delta_t}(\tilde{G}) = k$ and $L_{\tilde{\chi}}(k) = \alpha(X_t) = 1 - \max\{\mu(u, v) | u, v \in X_t\} = 1 - \delta_t$. It completes the proof. □

4. Some properties of fuzzy chromatic set of fuzzy graph

First, we investigate a property of δ-chromatic number of fuzzy graph. The property is presented in Theorem 4.1. After that, this property will be used to verify some properties of fuzzy chromatic set of fuzzy graph.

Theorem 4.1. *Let $\tilde{G}(V, \tilde{E})$ be a fuzzy graph and $\delta_1, \delta_2 \in [0, 1]$. The value $\delta_1 \geq \delta_2$ if and only if $\chi^{\delta_1}(\tilde{G}) \leq \chi^{\delta_2}(\tilde{G})$.*

Proof. Given $\delta_1, \delta_2 \in [0, 1]$. We consider all of partitions $\{S_1^{\delta_2}, ..., S_p^{\delta_2}\}$ of V which give $\chi^{\delta_2}(\tilde{G}) = p$. Assume that $\delta_1 \geq \delta_2$. We will construct a partition of V which gives $\chi^{\delta_1}(\tilde{G}) = q$ and $q \leq p$.

We consider the following three cases.

Case 1. There exist two sets $S_i^{\delta_2}$ and $S_j^{\delta_2}$ $(i \neq j)$ where $\mu(u, v) \leq \delta_1$ for all $u \in S_i^{\delta_2}$ and $v \in S_j^{\delta_2}$. In this case, we can construct a δ_1-fuzzy independent vertex set $S_i^{\delta_2} \cup S_j^{\delta_2}$. Without loss of generality, we put $S_i^{\delta_2}$ and $S_j^{\delta_2}$ in the right order such that $S_j^{\delta_2} = S_{i+1}^{\delta_2}$. Hence, we obtain a partition $\{W_1^{\delta_1}, ..., W_q^{\delta_1}\}$ where $W_1^{\delta_1} = S_1^{\delta_2}, W_2^{\delta_1} = S_2^{\delta_2}, \dots, W_i^{\delta_1} = S_i^{\delta_2} \cup S_{i+1}^{\delta_2}, W_{i+1}^{\delta_1} = S_{i+2}^{\delta_2}, W_{i+2}^{\delta_1} = S_{i+3}^{\delta_2}, \dots, W_q^{\delta_1} = S_p^{\delta_2}$ with $q < p$. Thus, $\chi^{\delta_1}(\tilde{G}) < \chi^{\delta_2}(\tilde{G})$.

Case 2. There exist $u \in S_i^{\delta_2}$ and $v \in S_j^{\delta_2}$ where $\mu(u, v) = \delta_1$, but there also exist $x \in S_i^{\delta_2}$ and $y \in S_j^{\delta_2}$ with $\mu(x, y) > \delta_1$. In this case, we can construct a δ_1-fuzzy independent vertex set $S_i^{\delta_2} \cup S_j^{\delta_2} - P$ where $P = \{x \in S_i^{\delta_2} | \mu(x, y) > \delta_1, y \in S_j^{\delta_2}\}$. Without loss of generality, we put $S_i^{\delta_2}$ and $S_j^{\delta_2}$ in the right order such that $S_j^{\delta_2} = S_{i+1}^{\delta_2}$. Hence, we obtain a partition $\{W_1^{\delta_1}, ..., W_q^{\delta_1}\}$ where $W_1^{\delta_1} = S_1^{\delta_2}, W_2^{\delta_1} = S_2^{\delta_2}, \dots, W_i^{\delta_1} = (S_i^{\delta_2} \cup S_{i+1}^{\delta_2}) - P, W_{i+1}^{\delta_1} = P \cup S_{i+2}^{\delta_2}, W_{i+2}^{\delta_1} = S_{i+3}^{\delta_2}, \dots, W_q^{\delta_1} = S_p^{\delta_2}$ with $q < p$. Thus, $\chi^{\delta_1}(\tilde{G}) < \chi^{\delta_2}(\tilde{G})$.

Case 3. There do not exist two sets $S_i^{\delta_2}$ and $S_j^{\delta_2}$ $(i \neq j)$ where $\mu(u, v) \leq \delta_1$ for all $u \in S_i^{\delta_2}$ and $v \in S_j^{\delta_2}$. In this case, we cannot construct a δ_1-fuzzy independent vertex set $S_i^{\delta_2} \cup S_j^{\delta_2}$. Hence, we obtain a partition $\{W_1^{\delta_1}, ..., W_q^{\delta_1}\}$ where $W_i^{\delta_1} = S_i^{\delta_2}$ for all $i = 1, 2, \dots, p$. Thus, $\chi^{\delta_1}(\tilde{G}) = \chi^{\delta_2}(\tilde{G})$.

Conversely, There are partition $W = \{W_1^{\delta_1}, ..., W_q^{\delta_1}\}$ such that $\chi^{\delta_1}(\tilde{G}) = q$ and partition $S = \{S_1^{\delta_2}, ..., S_p^{\delta_2}\}$ Such that $\chi^{\delta_2}(\tilde{G}) = p$. Assume that $q \leq p$. We will prove $\delta_1 \geq \delta_2$ by considering two cases as follows.

Case 1. $q < p$.
Since $q < p$, there exists a set $W_t^{\delta_1}$ in W such that $W_t^{\delta_1} = S_i^{\delta_2} \cup S_j^{\delta_2}$ for any $S_i^{\delta_2}, S_j^{\delta_2} \in S$ where $\mu(x, y) \leq \delta_1$ for all $x, y \in W_t^{\delta_1}$. Suppose that $\delta_1 < \delta_2$. It is obvious that $\mu(x, y) \leq \delta_1 < \delta_2$ for all $x, y \in S_i^{\delta_2} \cup S_j^{\delta_2}$. We put $S_i^{\delta_2}$ and $S_j^{\delta_2}$ in the right order such that $S_j^{\delta_2} = S_{i+1}^{\delta_2}$. Hence, we obtain a partition $S' = \{S_1^{\delta_2}, S_2^{\delta_2}, \dots, S_i^{\delta_2} \cup S_{i+1}^{\delta_2}, S_{i+2}^{\delta_2}, \dots, S_{p'}^{\delta_2}\}$ Such that $\chi^{\delta_2}(\tilde{G}) = p' < p$. It is a contradiction. Thus, $\delta_1 \geq \delta_2$.

Case 2. $q = p$.
In this case, there are two possibilities as follows.

1) $S_i^{\delta_2} = W_i^{\delta_1}$ for all $i = 1, 2, ..., p$. Let u, v be any elements of $S_i^{\delta_2}$. By Definition 2.8, $\mu(u, v) \leq \delta_2$. Since u and v are also elements of $W_i^{\delta_1}$, $\mu(u, v) \leq \delta_1$. Hence, $\delta_1 = \delta_2$.
2) There is at least a set $S_i^{\delta_2}$ in S such that $S_i^{\delta_2} \neq W_i^{\delta_1}$ for any $W_i^{\delta_1}$ in W. It means that there are $x, y \in W_i^{\delta_1}$ such that $x, y \notin S_i^{\delta_2}$. Consequently, $\delta_2 < \mu(x, y) \leq \delta_1$. Thus, $\delta_1 \geq \delta_2$. The theorem is proved.

Based on the new definition of fuzzy chromatic set of fuzzy graph, some properties of the fuzzy chromatic set are investigated. The results are presented in Theorem 4.2 and Theorem 4.3, respectively.

Theorem 4.2. *Let $\tilde{G}(V, \tilde{E})$ be a fuzzy graph and $\delta \in [0, 1]$. The δ-chromatic number 1 has the grade of membership $L_{\tilde{\chi}}(1) = 0$ if and only if $\exists u, v \in V$ such that $\mu(u, v) = 1$.*

Proof. Assume that there exists $\delta \in [0, 1]$ such that $\chi^{\delta}(\tilde{G}) = 1$ and $L_{\tilde{\chi}}(1) = 0$. Suppose that $\forall u, v \in V$, the grade of membership $\mu(u, v) < 1$. In other words, $\max\{\mu(u, v) | u, v \in V\} < 1$. Put $\delta = \max\{\mu(u, v) | u, v \in V\} < 1$. Because of the possible partition which gives $\chi^{\delta}(G) = 1$ is only $\{S^{\delta}\}$ where $S^{\delta} = V$, the grade of membership $L_{\tilde{\chi}}(1)$ is not equal to zero since $\delta < 1$. It is a contradiction. Thus there exist $u, v \in V$ such that $\mu(u, v) = 1$.

Conversely, assume that $\tilde{G}$ has at least two vertices $u, v \in V$ such that $\mu(u, v) = 1$.
Put $\delta = \max\{\mu(u, v) | u, v \in V\} = 1$. We have a δ-fuzzy independent vertex set $S^{\delta} = V$ such that $\chi^{\delta}(\tilde{G}) = 1$ and $L_{\tilde{\chi}}(1) = \max\{1 - \delta | \delta \in [0, 1], \chi^{\delta}(G) = 1\} = 0$. It completes the proof. □

Theorem 4.3. *Given a fuzzy graph $\tilde{G}(V, \tilde{E})$ with n vertices. Let $G^*(V^*, E^*)$ be the underlying crisp graph of $\tilde{G}$. Let C be a set of δ chromatic numbers of $\tilde{G}$ for all $\delta \in [0, 1]$. The fuzzy chromatic set $\tilde{\chi}(\tilde{G})$ has the following properties:*

i) *the closure of support $\tilde{\chi}(\tilde{G})$ is subset of C and it is finite,*
ii) *there exists $k_0 \in C$ such that $L_{\tilde{\chi}}(k_0) = 1$ (Normality),*
iii) *$L_{\tilde{\chi}}(i) \leq L_{\tilde{\chi}}(j)$ for any $i, j \in C$ with $i \leq j \leq k_0$,*
iv) *$L_{\tilde{\chi}}(i) = L_{\tilde{\chi}}(j) = 1$ for any $i, j \in C$ with $k_0 \leq i \leq j < n$.*

Proof. Let C be a set of δ-chromatic numbers of $\tilde{G}$ for all $\delta \in [0, 1]$.
i). The closure of support $\tilde{\chi}(\tilde{G})$ is the set $\{k \in C | L_{\tilde{\chi}}(k) > 0\}$. Obviously, it is subset of C. Since

$k \in \{1, \ldots, n\}$, the closure of support $\tilde{\chi}(\tilde{G})$ is finite.
ii). By Remark 3.1.1, there exists a δ-chromatic number $k_0 \in C$ where $k_0 = \chi^0(\tilde{G})$ with $L_{\tilde{\chi}}(k_0) = 1$.

iii). Let $A = \{\delta_a \in [0, 1] | \chi^{\delta_a}(\tilde{G}) = i\}$ and $B = \{\delta_b \in [0, 1] | \chi^{\delta_b}(\tilde{G}) = j\}$. Since $i \le j$, it follows from Theorem 4.1 that $\delta_a \ge \delta_b$ for all $\delta_a \in A$ and $\delta_b \in B$. This implies, $\min\{\delta_a | \delta_a \in A\} \ge \min\{\delta_b | \delta_b \in B\}$. Hence, $\max\{1 - \delta | \delta \in A\} \le \max\{1 - \delta | \delta \in B\}$. Thus, $L_{\tilde{\chi}}(i) \le L_{\tilde{\chi}}(j)$ for any $i, j \in C$ with $i \le j \le k_0$.

iv). By property $ii)$, $L_{\tilde{\chi}}(k_0) = 1$. By Remark 3.1.2, we have $L_{\tilde{\chi}}(n) = 1$. Since $k_0 \le i \le j < n$ then $1 \le L_{\tilde{\chi}}(i) \le L_{\tilde{\chi}}(j) \le 1$. Thus, $L_{\tilde{\chi}}(i) = L_{\tilde{\chi}}(j) = 1$. □

In other words, property $iv)$ state that If $\chi^\delta(\tilde{G}) = i$ has the degree $L_{\tilde{\chi}}(i) = 1$ and $i \ne n$ then $L_{\tilde{\chi}}(k) = 1$ for all $i \le k < n$.

We can see that properties $i) - iv)$ in Theorem 4.3 are the properties of a discrete fuzzy number. It can be concluded that the fuzzy chromatic set of fuzzy graph as given in Definition 3.1 is a discrete fuzzy number. In the next section, we call the fuzzy chromatic set by fuzzy chromatic number.

5. An algorithm for determining fuzzy chromatic number of fuzzy graph

We propose a fuzzy chromatic algorithm to determine fuzzy chromatic number of fuzzy graph. The fuzzy chromatic algorithm is based on δ-chromatic number of fuzzy graph, as given in Definition 3.1. The first step is to find δ-chromatic number of fuzzy graph for all $\delta \in [0, 1]$, and the second step is to determine fuzzy chromatic number of fuzzy graph. Because of δ-chromatic number is crisp number, we may use any algorithm for the chromatic number of crisp graph. Therefore, we derive the first step in the fuzzy chromatic algorithm by developing the algorithm which was introduced by Kishore and Sunitha [2].

The fuzzy chromatic algorithm

Let $G = (V, \tilde{E})$ be a fuzzy graph with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and the fuzzy edge set $\tilde{E}$. Let $f = \{\delta \in [0, 1] | \mu(v_l, v_m) = \delta, v_l, v_m \in V\}$.

Step 1: Initialize $\delta = \delta_0 = 0$.
Step 2: Initialize $k' = 0$.
Step 3: Find a family of δ-fuzzy independent vertex sets $\Gamma = \{S_1^\delta, \ldots, S_t^\delta\}$ where $S_i^\delta = \{v_l \cup v_m | \mu(v_l, v_m) \le \delta, l \ne m\}$. $i = 1, 2, \ldots, t$.
Step 4: Choose a subfamily $\Gamma' = \{S_1^\delta, \ldots, S_k^\delta\}$ such that $S_i^\delta \cap S_j^\delta = \emptyset, i \ne j, i, j = 1, \ldots k$ and $\Sigma_{j=1}^k |S_j^\delta|$ is maximum. Let $S = V - \bigcup_{j=1}^k S_j^\delta$.
Step 5: If $|S| = 0$, then:
If $k = 1$, then $\chi^\delta(\tilde{G}) = 1$ and go to Step 7,
else if $1 < k \le n$ then $\chi^\delta(\tilde{G}) = k$ and go to Step 7.
Step 6: If $|S| \ne 0$, then $\chi^\delta(\tilde{G}) = k$.
Let $k' = \chi^\delta(\tilde{G}) + k'$.
Put $\tilde{G} = \langle S \rangle$ and go to Step 3.
Step 7: $\chi^\delta(\tilde{G}) = k + k'$.
Step 8: For $0 < \delta_{i-1} \le \delta < \delta_i, \delta_i \in f$ repeat Step 2 until Step 7. If $\delta = \delta_m$ where $\delta_m = \max\{\delta | \delta \in f\}$, then stop and go to Step 9.
Step 9: Put $g_k = \{\delta \in f | \chi^\delta(G) = k\}$. For $k = 1, \ldots, n$, determine $L_{\tilde{\chi}}(k) = \max\{1 - \delta | \delta \in g_k\}$.
If $k = n$, then $L_{\tilde{\chi}}(n) = 1$,
If $\chi^\delta(\tilde{G}) = i$ has $L_{\tilde{\chi}}(i) = 1$ and $i \ne n$ then $L_{\tilde{\chi}}(k) = 1$ for all $i \le k < n$.
Thus, we obtain $\tilde{\chi}(\tilde{G}) = \{(k, L_{\tilde{\chi}}(k))\}$.
Step 10: End

Further, an example is given to illustrate the comparison between the fuzzy chromatic algorithm and the existing algorithm [10].

Example 5.1. Consider a fuzzy graph $\tilde{G}(V, \tilde{E})$ given in Fig.1. Firstly, we determine fuzzy chromatic number of $\tilde{G}$ by the existing algorithm as given in [10]. The first step is to find all of maximal fuzzy independent vertex sets X of $\tilde{G}$ and the second step is to select k-maximal fuzzy independent vertex sets with the maximum value of separation degree.

The fuzzy graph $\tilde{G}(V, \tilde{E})$ in Fig.1 has 13 maximal fuzzy independent vertex sets $X_1, X_2, \ldots, X_{13}$ with the degrees of independence $\alpha(X_i)$, $i = 1, \ldots, 13$. The task for coloring of the fuzzy graph $\tilde{G}$ with k colors is a task to select k-maximal fuzzy independent vertex sets

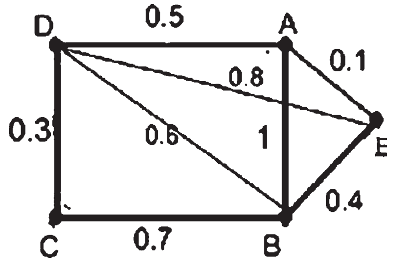

Fig. 1. A fuzzy graph $\tilde{G}(V, \tilde{E})$

Table 2
The Maximal fuzzy independent vertex sets of $\tilde{G}$ in Fig.1

	X_1	X_2	X_3	X_4	X_5	X_6	X_7
A	0	0	0	0.9	0.5	0	0
B	0.4	0.6	0	0	0	0.3	0.3
C	0	0	0.7	0.9	0.5	0.3	0.3
D	0.4	0	0.7	0	0.5	0.3	0
E	0	0.6	0	0.9	0	0	0.3

	X_8	X_9	X_{10}	X_{11}	X_{12}	X_{13}
A	0.2	0	0	1	0	0
B	0	0.2	0	0	1	0
C	0.2	0.2	1	1	0	0
D	0.2	0.2	0	0	0	1
E	0.2	0.2	1	0	0	0

$X_1, \ldots, X_k$ which give the maximum value of separation degree $L_{\tilde{\gamma}}(k)$.
For $k = 4$: coloring of $\tilde{G}$ with 4 colors is given by the sets $X_{10}, X_{11}, X_{12}, X_{13}$, where the maximum value of separation degree is $L_{\tilde{\gamma}}(4) = \min\{1, 1, 1, 1\} = 1$.
For $k = 3$: coloring of $\tilde{G}$ with 3 colors is given by the sets X_4, X_{12}, X_{13}, where the maximum value of separation degree is $L_{\tilde{\gamma}}(3) = 0.9$.
For $k = 2$: coloring of $\tilde{G}$ with 2 colors is given by the sets X_2, X_5, where the maximum value of separation degree is $L_{\tilde{\gamma}}(2) = 0.5$.
For $k = 1$: coloring of $\tilde{G}$ with 1 color is given by the set $X_{14} = \{A, B, C, D, E\}$, where the maximum value of separation degree is $L_{\tilde{\gamma}}(1) = 0$.
Thus, the fuzzy chromatic number of $\tilde{G}$ is

$$\tilde{\chi}(\tilde{G}) = \{(1, 0), (2, 0.5), (3, 0.9), (4, 1), (5, 1)\}$$

It is clear that the first step of the existing algorithm [10] is not a simple step, since we have to find all of subset of vertices $X \subseteq V$ with the degree of independence $\alpha(X)$ such that $\alpha(X^{'}) < \alpha(X)$ for any $X^{'} \supset X$. Therefore, the two steps of the existing algorithm are time consuming works.

Secondly, we use the fuzzy chromatic algorithm to determine fuzzy chromatic number of the fuzzy graph $\tilde{G}$. The first step is to find δ-chromatic number for all $\delta \in [0, 1]$. The result is presented in Table 3.
The second step of the fuzzy chromatic algorithm is to determine the fuzzy chromatic number $\tilde{\chi}(\tilde{G}) = \{(k, L_{\tilde{\chi}}(k)) | k = 1, 2, \ldots, n\}$, where the separation degree $L_{\tilde{\chi}}(k)$ for $k = 1, 2, \ldots, n$, are determined as follows.
Let $f = \{\delta \in [0, 1] | \mu(v_l, v_m) = \delta, v_l, v_m \in V\} = \{0.1, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 1\}$. We obtain:

Table 3
The Subfamilies of fuzzy independent vertex sets and the δ-chromatic number of $\tilde{G}$ for all $\delta \in [0, 1]$

δ_i	The Subfamilies Γ'	$\chi^{\delta_i}(\tilde{G})$
$0 \leq \delta < 0.1$	$\{\{C, E\}, \{A\}, \{\{B\}, \{D\}\}\}$,	4
	$\{\{A, C\}, \{B\}, \{D\}, \{E\}\}$	
$0.1 \leq \delta < 0.3$	$\{\{A, C, E\}, \{B\}, \{D\}\}$	3
$0.3 \leq \delta < 0.4$	$\{\{A, C, E\}, \{B\}, \{D\}\}$,	3
	$\{\{C, D\}, \{B\}, \{A, E\}\}$	
$0.4 \leq \delta < 0.5$	$\{\{A, C, E\}, \{B\}, \{D\}\}$,	3
	$\{\{C, D\}, \{B\}, \{A, E\}\}$,	
	$\{\{C, D\}, \{B, E\}, \{A\}\}$,	
	$\{\{A, C\}, \{B, E\}, \{D\}\}$	
$0.5 \leq \delta < 0.6$	$\{\{A, C, D\}, \{B, E\}\}$	2
$0.6 \leq \delta < 0.7$	$\{\{A, C, D\}, \{B, E\}\}$,	2
	$\{\{B, D\}, \{A, C, E\}\}$,	
$0.7 \leq \delta < 0.8$	$\{\{A, C, D\}, \{B, E\}\}$	2
	$\{\{B, D\}, \{A, C, E\}\}$,	
	$\{\{B, C, D\}, \{A, E\}\}$,	
	$\{\{B, C, E\}, \{A, D\}\}$,	
$0.8 \leq \delta < 1$	$\{\{A, C, D\}, \{B, E\}\}$	2
	$\{\{B, D\}, \{A, C, E\}\}$,	
	$\{\{B, C, D\}, \{A, E\}\}$,	
	$\{\{B, C, E\}, \{A, D\}\}$,	
	$\{\{B, C\}, \{A, E, D\}\}$,	
	$\{\{B, E, D\}, \{A, C\}\}$,	
1	V	1

$g_1 = \{\delta \in f | \chi^{\delta}(G) = 1\} = \{1\}$,
$g_2 = \{\delta | 0.5 \leq \delta < 1, \chi^{\delta}(G) = 2\}$,
$g_3 = \{\delta | 0.1 \leq \delta < 0.5, \chi^{\delta}(G) = 3\}$,
$g_4 = \{\delta | 0 \leq \delta < 0.1, \chi^{\delta}(G) = 4\}$,
$L_{\tilde{\chi}}(k) = \max\{1 - \delta | \delta \in g_k\}$.
We obtain:
$L_{\tilde{\chi}}(1) = 0$, $L_{\tilde{\chi}}(2) = 1 - 0.5 = 0.5$,
$L_{\tilde{\chi}}(3) = 1 - 0.1 = 0.9$,
$L_{\tilde{\chi}}(4) = 1$, and $L_{\tilde{\chi}}(5) = 1$.
Thus, the fuzzy chromatic number of $\tilde{G}(V, \tilde{E})$ is

$$\tilde{\chi}(\tilde{G}) = \{(k, L_{\tilde{\chi}}(k))\} = \{(1, 0), (2, 0.5), (3, 0.9), (4, 1), (5, 1)\}.$$

Since δ-chromatic number of fuzzy graph is crisp number, the algorithm for finding δ-chromatic number of fuzzy graph can be derived from the algorithm for the chromatic number of crisp graph. Hence, the step for finding δ-chromatic number of fuzzy graph is not time consuming work and it is easier than the step of finding all of maximal fuzzy independent vertex sets as given in the existing algorithm [10]. We conclude that the fuzzy chromatic algorithm becomes simpler than the existing algorithm [10].

6. Application

In this section, we give an application of the fuzzy chromatic number of fuzzy graph for the problem of medical virus infection on some locations (villages,towns). We assume that there are some virus-infected locations. The virus can propagate from one location to another location through some ways. If there are some individuals from one location who are interact with individuals from another location then the situation can be dangerous, that is they can propagate the virus. Since then, we have to protect the locations from the virus propagation. If we protect each location one by one, then it is a time consuming work and is needed much cost. Therefore, the problem is to determine the minimum number of group needed to isolate the locations such that the prevention of virus propagation can be maximized.

The locations can be represented by the vertices and the ways of virus propagation from different locations can be represented by the edges. Therefore, we can model the situation by a graph and the problem of determining the minimum group to isolate (protect) some locations can be regarded as the problem of determining the chromatic number of a graph . However if the situation is modeled by a crisp graph, then there are only two choices: the virus can propagate from one location to another location or the virus cannot propagate. In the real life problem, the situation between two locations may be not exactly known. Since then, the problem cannot be modeled by a crisp graph. In this paper, we deal with these non-deterministic phenomena by the fuzzy set theory. Thereby, we handle the situation by a fuzzy graph which based on fuzzy set theory.

A fuzzy graph $\tilde{G}(V, \tilde{E})$ is used to model the problem of medical virus infection. The vertices represent the locations and the edge (v_i, v_j) represents the way of virus propagation. Meanwhile the degree of membership $\mu(v_i, v_j)$ represents the possibility that the virus propagate through the way $v_i - v_j$. We solve the problem by using the fuzzy chromatic number $\tilde{\chi}(\tilde{G}) = \{(k, L_{\tilde{\chi}}(k))|k = 1, 2, \ldots, n\}$ where the number k indicates the minimum number of group needed to isolate the locations and the degree $L_{\tilde{\chi}}(k)$ indicate the possibility that the prevention of virus propagation can be maximized.

Example 2. Consider the fuzzy graph $\tilde{G}(V, \tilde{E})$ given in Fig.1. The vertex set V represents the set of locations $\{A, B, C, D, E\}$. There are 8 edges with the degrees of membership 0.1, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, respectively. The edge (v_i, v_j) represents the way of virus propagation and the degree of membership $\mu(v_i, v_j)$ represents the possibility that the virus propagate through the way $v_i - v_j$. From the previous example, we obtain the fuzzy chromatic number of $\tilde{G}$ is $\tilde{\chi}(\tilde{G}) = \{(1, 0), (2, 0.5), (3, 0.9), (4, 1), (5, 1)\}$, which can be interpreted as follows.

1) If we only use one group, that is $\{A, B, C, D, E\}$, it means that we do not isolate the locations. Hence, the possibility that we can prevent the virus propagation is 0.
2) If we isolate the locations by 2 groups, then the possibility that we can prevent the virus propagation is 0.5. We can isolate the locations $\{A, C, D\}$ in the first group and isolate the locations $\{B, E\}$ in the second group. There are some choices to isolate the locations by the two groups, which are listed in Table 3 (column 2).
3) If we isolate the locations by 3 groups, then the possibility that we can prevent the virus propagation is 0.9. We can isolate: the locations $\{A, C, E\}$ in the first group, the location $\{B\}$ in the second group, and the location $\{D\}$ in the third group. There are some choices to isolate the locations by the three groups, which are listed in Table 3 (column 2).
4) If we isolate some locations by 4 or 5 groups, then the possibility that we can prevent the virus propagation is 1. It means that the condition is very safe. The 4 or 5 groups for isolating the locations are listed in Table 3 (column 2).

7. Conclusions

In this paper, we initialize a new approach to determine fuzzy chromatic number of fuzzy graph with crisp vertex set and fuzzy edge set. The new approach is based on δ-chromatic number of fuzzy graph. Some properties of the fuzzy chromatic number of fuzzy graph are also investigated. We prove that the fuzzy chromatic number which is resulted by the new approach is equivalent to the fuzzy chromatic number as given in [10].

Furthermore, we propose a fuzzy chromatic algorithm to determine fuzzy chromatic number of fuzzy graph based on the new approach. The first step of the fuzzy chromatic algorithm is to find δ-chromatic number of fuzzy graph for all $\delta \in [0, 1]$, and the second step is to determine the fuzzy chromatic number of fuzzy graph. In the first step, the coloring problem of fuzzy graph is transformed into the crisp coloring. Hence, the second step of the fuzzy chromatic algorithm becomes simpler and easier to apply than the existing algorithm [10]. Further, the fuzzy chromatic number is used to solve the problem of determining the minimum number of group needed to protect some locations such that the prevention of virus propagation can be maximized.

In our future work, we plan to extend the concepts of δ-fuzzy independent vertex set, δ-chromatic number, and fuzzy chromatic number of second kind fuzzy graph, that is a fuzzy graph with fuzzy vertex set and fuzzy edge set. After that, we will employ the current results to construct an algorithm for determining fuzzy chromatic number of the second kind fuzzy graph.

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