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On total vertex irregularity strength of generalized uniform cactus chain graphs with pendant vertices

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Abstract

In this research, we use the concept of a vertex irregular total k labeling and tvs(G). Specifically, we investigate tvs of generalized uniform cactus chain graphs $C_r(C_n^{n-2})$ with $(n-2)r$ pendant vertices and length r. The result is as follows: $tvs(C_r(C_n^{n-2}) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil$ for $n \geq 6$.

Subject Classification: *05C78.*

Keywords: *Vertex irregular total k-labeling, Tvs, Uniform, Cactus chain.*

1. Introduction

Throughout this paper, we consider a graph $G(V, E)$ with $V = V(G)$ and $E = E(G)$. It is presumed that G is finite, simple, and undirected. “A total labeling of G is a function λ that assigns $V \cup E$ into a set of integers,

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mentioned as labels" [9,13,17,18]. "The function λ is also called a vertex irregular total k-labeling if the weights $wt_\lambda(u) \neq wt_\lambda(v)$ for $u \neq v \in V$ where $wt_\lambda(u) = \lambda(u) + \sum_{ux \in E(G)} \lambda(ux)$. Moreover, a total vertex irregularity strength of G, symbolized by tvs(G), is a minimum number k for which G has a vertex irregular total k-labeling" [2]. Lower bound of tvs of a graph G was proposed in [5]: $\left\lceil \frac{(p+\delta)}{(\Delta+1)} \right\rceil \leq tvs(G) \leq p + \Delta - 2\delta + 1$, where p is |V| $\delta = \min\{d(v) \mid \forall v \in V\}$ and $\Delta = \max\{d(u) \mid \forall u \in V\}$. Furthermore, another lower bound was given by Anholcer, et al. in [3]. In addition, tvs of any connected graph which consists of n_i vertices having degree i for $\delta \leq i \leq \Delta$ satisfies lower bound [14]:

$$tvs \geq \max\left\{ \left\lceil \frac{\delta(G) + n_{\delta(G)}}{\delta(G)+1} \right\rceil, \left\lceil \frac{\delta(G) + n_{\delta(G)} + n_{\delta(G)+1}}{\delta(G)+2} \right\rceil, \ldots, \left\lceil \frac{\delta(G) + \sum_{i=\delta(G)}^{\Delta(G)} n_i}{\Delta(G)+1} \right\rceil \right\} \quad (1)$$

Many scholars have invented tvs of many graph classes such as in [1,2,3,6,11,14], etc. More results on tvs of any graphs can be seen in [6] and [8]. The concept of cactus graph and some results on the cactus can be found in [4,7,9,12,19], etc. "A cactus graph is defined as a connected graph for which each edge lies in exactly one cycle. In other words, the cactus graph comprises several blocks where each block is either a cycle or an edge. If each block of the cactus graph has at most two cut-vertices and every two blocks share exactly one cut vertex, then we call it as a cactus chain graph" [7,12]. Some results related to tvs of cactus chain graphs have been invented in [15,16]. In this article, we examine tvs of generalized uniform cactus chain graphs $C_r(C_n^{n-2})$ with $(n-2)r$ pendant vertices and length r.

2. Main Result and Discussion

We provide formulas for labels of vertices and edges and a formula for tvs of generalized uniform cactus chains $C_r(C_n^{n-2})$.

2.1 *Generalized uniform cactus chain graphs with pendant vertices*

Firstly, we present the concept of generalized uniform cactus chains having pendant vertices in Definition 2.1.1.

Definition 2.1.1 : "A cactus graph, in which each block is a cycle with the same size n, will be called as an n-uniform cactus graph for any natural number n. If each cycle of the n-uniform cactus has at most two cut-vertices and each cut-vertex is shared by exactly two cycles, then it is called an n-uniform cactus chain graph. The number of cycles indicates the length of the cactus chain graph" [7,12]. Further, generalized uniform cactus chain graphs having pendant vertices of length r, denoted by $C_r(C_n^{n-2})$, are the n-uniform cactus chains in which each block is a cycle C_n connected with n-2 pendant vertices. The vertex and edge sets of $C_r(C_n^{n-2})$ are as follows:

$$V(C_r(C_n^{n-2})) = \{a_i, b_{1i}, b_{2i}, b_{3i}, b_{4i}, \ldots, b_{pi}, c_{1i}, c_{2i}, c_{3i}, c_{4i}, \ldots, c_{qi}, a_{i+1}\} \cup \{b_{1i}', b_{2i}', b_{3i}', b_{4i}', \ldots, b_{pi}', c_{1i}', c_{2i}', c_{3i}', c_{4i}', \ldots, c_{qi}'\}$$

and

$$E(C_r(C_n^{n-2})) = \left\{ \begin{array}{l} a_i b_{1i}, b_{1i} b_{2i}, b_{2i} b_{3i}, b_{3i} b_{4i}, \ldots, b_p a_{i+1}, a_i c_{1i}, c_{1i} c_{2i}, c_{2i} c_{3i}, c_{3i} c_{4i}, \ldots, c_q a_{i+1}, \\ b_{1i} b_{1i}', b_{2i} b_{2i}', b_{3i} b_{3i}', b_{4i} b_{4i}', \ldots, b_p b_p', c_{1i} c_{1i}', c_{2i} c_{2i}', c_{3i} c_{3i}', c_{4i} c_{4i}', \ldots, c_q c_q' \end{array} \right\}$$

$\forall i = 1,2,3,\ldots,r,$ where r indicates the length of the chain graphs and the indexes p and q are equal to $\frac{n-2}{2}$ if n is even. Meanwhile, $p = \left\lceil \frac{n-2}{2} \right\rceil$ and $q = \left\lfloor \frac{n-2}{2} \right\rfloor$ for odd n.

2.2 *Tvs of the generalized uniform cactus chain graphs*

Theorem 2.2.1 : *Given the generalized uniform cactus chain graphs $C_r(C_n^{n-2})$ with $(n-2)r$ pendant vertices and length $r \geq 2$. Then,* $tvs(C_r(C_n^{n-2})) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil, n \geq 6.$

Proof: Let us consider the vertices b_{ji}' and c_{li}' with degree 1 for $i = 1,2,\ldots,r$, where the indexes of j and l are $j = 1,2,\ldots,\frac{(n-2)}{2}$; $l = 1,2,\ldots,\frac{(n-2)}{2}$ when n is even. Whereas, $j = 1,2,\ldots,\left\lceil \frac{(n-2)}{2} \right\rceil$; $l = 1,2,\ldots,\left\lfloor \frac{(n-2)}{2} \right\rfloor$ if n is odd. Further, b_{ji}, c_{li} are vertices of degree 3. Moreover, vertices a_1, a_{r+1} have degree 2; and a_{i+1} have degree 4 for $i = 1,2,\ldots,r-1$. According to Lower bound (1), we get

$$tvs(C_r(C_n^{n-2})) \geq \max \left\{ \begin{array}{l} \left\lceil \frac{1+(n-2)r}{2} \right\rceil, \left\lceil \frac{(n-2)r+3}{3} \right\rceil, \left\lceil \frac{(n-2)r+3+(n-2)r}{4} \right\rceil, \\ \left\lceil \frac{2(n-2)r+3+(r-1)}{5} \right\rceil \end{array} \right\} \geq$$

$$\left\lceil \frac{2(n-2)r+3}{4} \right\rceil \tag{2}$$

To proof that $k = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil$ is the upper bound of the tvs, we establish a total k-labeling $f : V \cup E \rightarrow \{1, 2, \ldots, k\}$ and show that it is vertex irregular by means of two cases.

Case 1 : For n is odd ($n \geq 7$)

Case 1 is divided into two subcases as follows.

Subcase 1.1 : When the length r is odd.

In this subcase, we provide labels $f(x)$ and $f(e)$ for each $x \in V(C_r(C_n^{n-2}))$ and $e \in E(C_r(C_n^{n-2}))$ in Table 1.

Table 1

Labels of vertices and edges of $C_r(C_n^{n-2})$ for both n and r are odd

x, e	$f(x)$ and $f(e)$	The index $i = 1, 2, \ldots, r$
b_{ji}	$\left\lfloor \frac{n-2}{2} \right\rfloor i - \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor - (j+1) \right\}$,	$j = 1, 2, \ldots, \left\lceil \frac{n-2}{2} \right\rceil$ (3)
b_{ji}'	$\left\lfloor \frac{n-2}{2} \right\rfloor i - \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor - j \right\}$,	j as in (3)
c_{ji}	$\left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor i - \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor - (j+1) \right\}$,	$j = 1, 2, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor$ (4)
c_{ji}'	$\left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor i - \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor - j \right\}$,	j as in (4)
a_i	i	
a_{r+1}	1	
$a_i b_{1i}$	$\left\lceil \frac{2(n-2)r+3}{4} \right\rceil$	j equals to (3)
$b_{ji} b_{(j+1)i}$	$\left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1$	$j = 1, 2, \ldots, \left\lceil \frac{n-2}{2} \right\rceil - 1$
$b_{\lceil \frac{n-2}{2} \rceil i} a_{i+1}$	$\left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1$	

Contd...

$a_i c_{1i}$	$\left\lceil \frac{2(n-2)r+3}{4} \right\rceil$	
$c_{ji} c_{(j+1)i}$	$\left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1$	$j = 1, 2, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor - 1$
$c_{\left\lfloor \frac{n-2}{2} \right\rfloor i} a_{i+1}$	$\left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1$	$i = 1, 2, \ldots, r-1$
$c_{\left\lfloor \frac{n-2}{2} \right\rfloor r} a_{r+1}$	$\left\lceil \frac{2(n-2)r+3}{4} \right\rceil$	
$b_{ji} b_{ji}'$	i	j as in (3)
$c_{ji} c_{ji}'$	$\left\lceil \frac{2(n-2)r+3}{4} \right\rceil$	j equals to (4)

Under labeling f, we observe the vertex weights:

$$wt(b_{ji}) = 2\left\lceil \frac{2(n-2)r+3}{4} \right\rceil + \left\lceil \frac{n-2}{2} \right\rceil i - \left\{ \left\lceil \frac{n-2}{2} \right\rceil - j \right\},$$

$$wt(b_{ji}') = \left\lceil \frac{n-2}{2} \right\rceil i - \left\{ \left\lceil \frac{n-2}{2} \right\rceil - (j+1) \right\}, 1 \le i \le r, j = 1, 2, \ldots, \left\lceil \frac{n-2}{2} \right\rceil$$

$$wt(c_{ji}') = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil + \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor i - \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor - j \right\},$$

$$wt(c_{ji}) = 3\left\lceil \frac{2(n-2)r+3}{4} \right\rceil + \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor i - \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor - (j-1) \right\},$$

$$1 \le i \le r, \; j = 1, 2, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor$$

$$wt(a_1) = 2\left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1,$$

$$wt(a_{r+1}) = 2\left\lceil \frac{2(n-2)r+3}{4} \right\rceil,$$

$$wt(a_{i+1}) = 4\left\lceil \frac{2(n-2)r+3}{4} \right\rceil + (i-3), 1 \le i \le r-1.$$

Subcase 1.2 : When the length r is even.

In this subcase, we give labels $f(x)$ and $f(e)$ for each $x \in V(C_r(C_n^{n-2}))$ and $e \in E(C_r(C_n^{n-2}))$ in Table 2.

Table 2
Labels of vertices and edges of $C_r(C_n^{n-2})$ for *n* is odd and *r* is even

$f(x)$ and $f(e)$	The index $i = 1, 2, \ldots, r$
$f(b_{ji}) = \left\lfloor \frac{n-2}{2} \right\rfloor i - \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor - (j+2) \right\},$	$j = 1, 2, \ldots, \left\lceil \frac{n-2}{2} \right\rceil - 1$
$f\left(b_{\left\lceil \frac{n-2}{2} \right\rceil i}\right) = \left\lfloor \frac{n-2}{2} \right\rfloor i - \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor - \left(\left\lceil \frac{n-2}{2} \right\rceil + 2 \right) \right\},$	$i = 1, 2, \ldots, r;\ r \geq 4$ $i = 1\ (r = 2)$
$f\left(b_{\left\lceil \frac{n-2}{2} \right\rceil r}\right) = \left\lfloor \frac{n-2}{2} \right\rfloor r + 2$	$r = 2$
$f(b_{ji}') = \left\lfloor \frac{n-2}{2} \right\rfloor i - \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor - j \right\},$	j as in (3)
$f(c_{ji}) = \left(\frac{r}{2} \right) + \left\lfloor \frac{n-2}{2} \right\rfloor i - \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor - (j+2) \right\},$	$j = 1, 2, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor - 1$
$f\left(c_{\left\lfloor \frac{n-2}{2} \right\rfloor i}\right) = \left(\frac{r}{2} \right) + \left\lfloor \frac{n-2}{2} \right\rfloor i - \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor - \left(\left\lfloor \frac{n-2}{2} \right\rfloor + 2 \right) \right\},$	$i = 1, 2, \ldots, r-1$
$f\left(c_{\left\lfloor \frac{n-2}{2} \right\rfloor r}\right) = \left(\frac{r}{2} \right) + \left\lfloor \frac{n-2}{2} \right\rfloor r + 1$	
$f(c_{ji}') = \left(\frac{r}{2} \right) + \left\lfloor \frac{n-2}{2} \right\rfloor i - \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor - j \right\},$	j is equal to (4)
$f(a_1) = f(a_{r+1}) = 2$	$r \geq 4$
$f(a_{i+1}) = i + 3$	$i = 1, 2, \ldots, r-1;\ r \geq 4.$
$f(a_1) = 2; f(a_2) = 1$	$r = 2$
$f\left(a_i b_{1i}\right) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1,$	
$f(b_{ji} b_{(j+1)i}) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1,$	$j = 1, 2, \ldots, \left\lceil \frac{n-2}{2} \right\rceil - 1$

Contd...

$f\left(b_{\left\lceil \frac{n-2}{2} \right\rceil i} a_{i+1}\right) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1,$	$i = 1, 2, \ldots, r;\ r \geq 4$
$f\left(b_{\left\lceil \frac{n-2}{2} \right\rceil i} a_{i+1}\right) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1,$	$i = r-1 = 1;\ r = 2$
$f\left(b_{\left\lceil \frac{n-2}{2} \right\rceil r} a_{r+1}\right) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil,$	$r = 2$
$f(a_i c_{1i}) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1,$	
$f(c_{ji} c_{(j+1)i}) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1,$	$j = 1, 2, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor - 1$
$f\left(c_{\left\lfloor \frac{n-2}{2} \right\rfloor i} a_{i+1}\right) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1,$	$i = 1, 2, \ldots, r-1$
$f\left(c_{\left\lfloor \frac{n-2}{2} \right\rfloor r} a_{r+1}\right) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil,$	
$f(b_{ji} b_{ji}') = i,$	j is given in (3)
$f(c_{ji} c_{ji}') = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil,$	j is equal to (4)

We get the weights of vertices in the following way:

$$
\begin{aligned}
wt(b_{ji}) &= 2\left\lceil \frac{2(n-2)r+3}{4} \right\rceil + \left\lceil \frac{n-2}{2} \right\rceil i - \left\{ \left\lceil \frac{n-2}{2} \right\rceil - (j+1) \right\}, \\
wt(b_{ji}') &= \left\lceil \frac{n-2}{2} \right\rceil i - \left\{ \left\lceil \frac{n-2}{2} \right\rceil - (j+1) \right\}, 1 \leq i \leq r, j = 1, 2, \ldots, \left\lceil \frac{n-2}{2} \right\rceil \\
wt(c_{ji}') &= \left\lceil \frac{2(n-2)r+3}{4} \right\rceil + \frac{r}{2} + \left\lfloor \frac{n-2}{2} \right\rfloor i - \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor - j \right\}, \\
wt(c_{ji}) &= 3\left\lceil \frac{2(n-2)r+3}{4} \right\rceil + \frac{r}{2} + \left\lfloor \frac{n-2}{2} \right\rfloor i - \left\{ \left\lfloor \frac{n-2}{2} \right\rfloor - (j) \right\}, \\
& \quad 1 \leq i \leq r,\ j = 1, 2, \ldots, \left\lfloor \frac{n-2}{2} \right\rfloor \\
wt(a_1) &= 2\left\lceil \frac{2(n-2)r+3}{4} \right\rceil,
\end{aligned}
$$

$$wt(a_{r+1}) = 2\left\lceil \frac{2(n-2)r+3}{4} \right\rceil + 1, wt(a_{i+1}) = 4\left\lceil \frac{2(n-2)r+3}{4} \right\rceil + (i-1), 1 \le i \le r-1.$$

Case 2 : For n is even (n ≥ 6).

In this case, we construct vertex and edge labels as in Table 3.

Table 3

Vertex and edge-labels of $C_r(C_n^{n-2})$ for n is even.

$f(x)$ and $f(e)$	The index $1 \le i \le r$
$f(b_{ji}) = \left(\frac{n-4}{2}\right)i + j - \left\{\left(\frac{n-4}{2}\right) - 2\right\},$	$1 \le j \le \frac{n-2}{2}$ (5)
$f(b_{ji}') = \left(\frac{n-4}{2}\right)i + \left\{j - \left(\frac{n-4}{2}\right)\right\},$	j is equal to (5)
$f(c_{ji}) = \left(\frac{n-2}{2}\right)i + \left\{j - \left(\frac{n-6}{2}\right)\right\},$	$1 \le i \le r-1;\ 1 \le j \le \frac{n-2}{2} - 1$
$f(c_{jr}) = \left(\frac{n-2}{2}\right)r + 1,$	j is given in (5)
$f(c_{ji}') = \left(\frac{n-2}{2}\right)i + \left\{j - \left(\frac{n-2}{2}\right)\right\},$	j as in (5)
$f(a_{i+1}) = i + 3,$	$i = 1, 2, \ldots, r-1$
$f(a_1) = f(a_{r+1}) = 2,$	
$f(a_i b_{1i}) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1,$	
$f(b_{ji} b_{(j+1)i}) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1,$	$1 \le j \le \frac{n-2}{2} - 1$
$f\left(b_{\frac{n-2}{2}i} a_{i+1}\right) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1,$	
$f(a_i c_{1i}) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1,$	
$f(c_{ji} c_{(j+1)i}) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil - 1,$	$1 \le j \le \frac{n-2}{2} - 1$

Contd...

$f\left(c_{\frac{n-2}{2}i}a_{i+1}\right)=\left\lceil\frac{2(n-2)r+3}{4}\right\rceil-1,$	$i=1,2,\ldots,r-1$
$f\left(c_{\frac{n-2}{2}r}a_{r+1}\right)=\left\lceil\frac{2(n-2)r+3}{4}\right\rceil,$	
$f(b_{ji}b_{ji}')=i,$	j is given in (5)
$f(c_{ji}c_{ji}')=\left\lceil\frac{2(n-2)r+3}{4}\right\rceil$	j equals to (5)

In this case, the weights of vertices are provided below:

$$wt(b_{ji}) = 2\left\lceil\frac{2(n-2)r+3}{4}\right\rceil+\frac{n-2}{2}i+j-\left(\frac{n-4}{2}\right),$$

$$wt(a_1) = 2\left\lceil\frac{2(n-2)r+3}{4}\right\rceil,$$

$$wt(b_{ji}') = \frac{n-2}{2}i+j-\left(\frac{n-4}{2}\right), 1\le i\le r,\ j=1,2,\ldots,\frac{n-2}{2}$$

$$wt(c_{ji}') = \left\lceil(2(n-2)r+3)/4\right\rceil+\frac{n-2}{2}i+j-\frac{n-2}{2},$$

$$wt(c_{ji}) = 3\left\lceil(2(n-2)r+3)/4\right\rceil+\frac{n-2}{2}i+j-\frac{n-2}{2}, 1\le i\le r, j=1,2,\ldots,\frac{n-2}{2}$$

$$wt(a_{r+1}) = 2\left\lceil(2(n-2)r+3)/4\right\rceil+1,$$

$$wt(a_{i+1}) = 4\left\lceil(2(n-2)r+3)/4\right\rceil+(i-1), 1\le i\le r-1.$$

In Case 1 and Case 2, the minimum label of elements of $C_r(C_n^{n-2})$ is 1 and the largest integer of vertex and edge labels is $\left\lceil(2(n-2)r+3)/4\right\rceil$. It is shown that the vertex weights are all different and it proves the upper bound $tvs(C_r(C_n^{n-2}))\le\left\lceil(2(n-2)r+3)/4\right\rceil$. According to the Lower bound (2), it shows that $tvs(C_r(C_n^{n-2}))=\left\lceil(2(n-2)r+3)/4\right\rceil, n\ge 6$. The proof is complete. ■

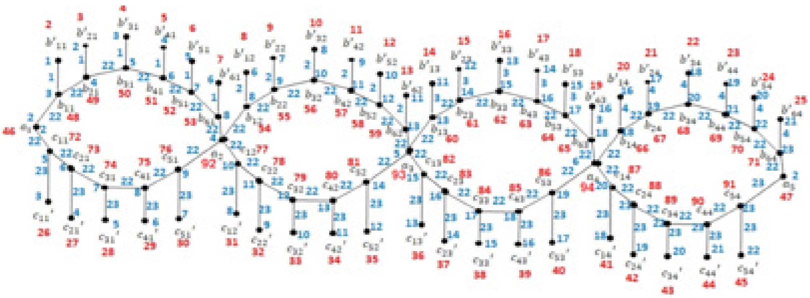

Fig. 1

The pattern to get $tvs(C_4(C_{13}^{11})) = 23$.

Example 2.1.1 : Figure 1 illustrates labels for vertices and edges such that $tvs(C_4(C_{13}^{11})) = 23$.

3. Conclusions

The formula for tvs of generalized uniform cactus chain graphs $C_r(C_n^{n-2})$ with $(n-2)r$ pendant vertices and length r has been proved in this article. We get $tvs(C_r(C_n^{n-2})) = \left\lceil \frac{2(n-2)r+3}{4} \right\rceil$ for $n \geq 6$. The patterns to get the vertex irregular total k-labeling were presented in the theorem. In upcoming research, we are interested to investigate tvs or tes of generalized tadpole chain graphs. Also, we will examine the tvs or tes using algorithmic approaches.

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