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Isnaini Rosyida , Eka Ningrum , Mulyono & Diari Indriati

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On the total edge irregularity strength of general uniform cactus chain graphs with pendant vertices

Isnaini Rosyida *
Eka Ningrum †
Mulyono §
Department of Mathematics
Faculty of Mathematics and Natural Sciences
Universitas Negeri Semarang
Semarang
Indonesia

Diari Indriati ‡
Faculty of Mathematics and Natural Sciences
Universitas Sebelas Maret
Surakarta
Indonesia

Abstract

Let $G(V,E)$ be a graph. Throughout this paper, we use the notions of edge irregular total k-labeling and total edge irregularity strength of G (tes (G)). We verify tes of general uniform cactus chain graphs $C_r(C_n^{n-2})$ having $(n-2)r$ pendant vertices and length r. The result obtained is as follows: $tes(C_r(C_n^{n-2}) = \left\lceil \frac{(2n-2)r+2}{3} \right\rceil$ for $n \geq 6$.

Subject Classification: *05C78.*

Keywords: *Edge irregular total k-labeling, Tes, Uniform, Cactus chain.*

*E-mail: iisisnaini@gmail.com; isnaini@mail.unnes.ac.id (corresponding author)
†E-mail: ekaningrum5@gmail.com
§E-mail: mulyono.mat@mail.unnes.ac.id
‡E-mail: diari_indri@yahoo.co.id

1. Introduction

We consider a graph $G(V, E)$ where $V = V(G)$ is a vertex set and $E = E(G)$ is an edge set. Suppose that graph G is simple, finite, and also undirected. A mapping λ from $V \cup E$ into a set {1, 2, ..., k} is called a total k-labeling of G [9,13,17,18]. The function λ is mentioned as an edge irregular total k-labeling if the edge-weights $wt_\lambda(pq) \neq wt_\lambda(rs)$ for all distinct edges $pq \neq rs \in E$ with $wt_\lambda(pq) = \lambda(p) + \lambda(pq) + \lambda(q)$. A total edge irregularity strength of G, tes (G), is a minimum number k so that there is an edge irregular total k-labeling of G. The bounds for tes of any graph are as follows [4]:

$$\left\lceil \frac{|E(G)|+2}{3} \right\rceil \leq tes(G) \leq |E|. \tag{1}$$

Some results of tes of various graph classes have been invented, such as in [1,3,5,7,10,11], etc. The readers may refer to [5] and [7] for more results on tes of any graphs.

The notion of cactus graph and several results related to the cactus can be seen in [2,6,8,12,19,20], etc. Some inventions of tes of cactus chains have been proposed in [14,15,16]. In this paper, we investigate tes of general uniform cactus chains $C_r(C_n^{n-2})$ having $(n-2)r$ pendant vertices and length r.

2. Main Result and Discussion

In this paper, we present definition and formula for tes of general uniform cactus chain graphs $C_r(C_n^{n-2})$.

2.1 *General uniform cactus chain graphs with pendant vertices*

The concept of cactus graph can be found in [6,12]. Meanwhile, the general uniform cactus chain graphs are defined as follows. "An n-uniform cactus graph is a cactus graph in which each block is a cycle with the same size n for any positive integer n. If each cycle of the n-uniform cactus has at most two cut-vertices and each cut-vertex is shared by exactly two cycles, then it is called an n-uniform cactus chain graph. The number of cycles indicates the length of the cactus chain graph" [15]. Furthermore, the general uniform cactus chain graphs with length r, $C_r(C_n^{n-2})$, are defined as the n-uniform cactus chains having r blocks where each block is in form of a cycle C_n connected with $n-2$ pendant vertices. The vertices and edges of $C_r(C_n^{n-2})$ are as follows:

$$V(C_r(C_n^{n-2})) = \{a_i, b_{1i}, b_{2i}, b_{3i}, b_{4i}, \dots, b_{pi}, c_{1i}, c_{2i}, c_{3i}, c_{4i}, \dots, c_{qi}, a_{i+1}\} \cup \{b_{1i}', b_{2i}', b_{3i}', b_{4i}', \dots, b_{pi}', c_{1i}', c_{2i}', c_{3i}', c_{4i}', \dots, c_{qi}'\}$$

and

$$E(C_r(C_n^{n-2})) = \left\{ \begin{matrix} a_i b_{1i}, b_{1i}b_{2i}, b_{2i}b_{3i}, b_{3i}b_{4i}, \dots, b_p a_{i+1}, a_i c_{1i}, c_{1i}c_{2i}, c_{2i}c_{3i}, c_{3i}c_{4i}, \dots, c_q a_{i+1}, \\ b_{1i}b'_{1i}, b_{2i}b'_{2i}, b_{3i}b'_{3i}, b_{4i}b'_{4i}, \dots, b_p b'_p, c_{1i}c'_{1i}, c_{2i}c'_{2i}, c_{3i}c'_{3i}, c_{4i}c'_{4i}, \dots, c_q c'_q \end{matrix} \right\}$$

$\forall i = 1,2,3,\dots,r,$ where r is the length of the chain graphs (the number of blocks in the chain) and the indexes p and q are defined as:

$$p = \begin{cases} \frac{n-2}{2}, & n \text{ is even} \\ \left\lceil \frac{n-2}{2} \right\rceil, & n \text{ is odd} \end{cases} \text{ and } q = \begin{cases} \frac{n-2}{2}, & n \text{ is even} \\ \left\lfloor \frac{n-2}{2} \right\rfloor, & n \text{ is odd} \end{cases}.$$

2.2 *Tes of general uniform cactus chain graphs with pendant vertices*

We prove the tes of general uniform cactus chains in this section.

Theorem 2.2.1 : *Let* $C_r(C_n^{n-2})$ *be general uniform cactus chain graphs having* $(n-2)r$ *pendant vertices,* $n \geq 6$, *and the length* $r \geq 2$. *Then, the tes is*

$$tes(C_r(C_n^{n-2})) = \left\lceil \frac{2(n-2)r+2}{3} \right\rceil. \quad (2)$$

Proof : Let b'_{ji} and c'_{li} be vertices of the general uniform cactus chains with degree 1 for $i = 1,2,\dots,r$. The indexes of j and l are $j = 1,2,\dots,\frac{(n-2)}{2}$; $l = 1,2,\dots,\frac{(n-2)}{2}$ for even number n. Further, $j = 1,2,\dots,\left\lceil \frac{(n-2)}{2} \right\rceil$; $l = 1,2,\dots,\left\lfloor \frac{(n-2)}{2} \right\rfloor$ for odd number n. Meanwhile, b_{ji}, c_{li} are the vertices of degree 3. Further, vertices a_1, a_{r+1} have degree 2; and a_{i+1} have degree 4 for $i = 1,2,\dots,r-1$.

Based on lower bound (1), we have

$$tes(C_r(C_n^{n-2})) \geq \left\lceil \frac{|E(C_r(C_n^{n-2}))|+2}{3} \right\rceil = \left\lceil \frac{2(n-2)r+2}{3} \right\rceil.$$

We verify the upper bound through 3 cases.

Case 1 : $n \equiv 1 \bmod 3$, $n \geq 7$.

In the first case, we give labels to vertices and edges as follows:

$$f(a_i)=\frac{1}{3}\{(2n-2)i-(2n-5)\};\ f(b_i)=\frac{1}{3}\{(2n-2)i-(2n-5)\}, i=1,2,\dots,r$$

$$f(b_{ji})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 1 \bmod 3,\ j\equiv 2 \bmod 3,\ 2\le j\le t-1$$

where $i=1,2,\dots,r;\ t=\left\lceil\frac{n-2}{2}\right\rceil$ if n is odd and $t=\frac{n-2}{2}$ if n is even

$$f(b_{ji})=\frac{1}{3}\{(2n-2)i-(2n-(2+4j))\}, j\equiv 0 \bmod 3,\ 3\le j\le t-1, i=1,2,\dots,r$$

$$f(b_{ti})=\frac{1}{3}\{(2n-2)i\},\ n \text{ is even or odd},\ i=1,2,\dots,r$$

$$f(a_{i+1})=\left\lceil\frac{(2n-2)i+2}{3}\right\rceil; f(b_{1i}{}')=\frac{1}{3}\left\{(2n-2)r-[2n-8]\right\}, i=1,2,\dots,r$$

$$f(b_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(2+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 1 \bmod 3,\ 4\le j\le t-1$$

$$f(b_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\}, j\equiv 2 \bmod 3,\ 2\le j\le t-1$$

$$f(b_{ji}{}')=\frac{1}{3}\{(2n-2)i-[2n-(2+4j)]\},\ j\equiv 0 \bmod 3,\ 3\le j\le t-1,\ i=1,2,\dots,r$$

$$f(b_{ti}{}')=\frac{1}{3}\{(2n-2)i\}\ (n \text{ is odd});\ f(b_{ti}{}')=\frac{1}{3}\{(2n-2)i\}-1\ (n \text{ is even})$$

$$f(c_{ji})=\frac{1}{3}\{(2n-2)i-[2n-(5+4j)]\},\ j\equiv 0 \bmod 3,\ 3\le j\le q-1, i=1,2,\dots,r$$

where $q=\left\lfloor\frac{n-2}{2}\right\rfloor$ if n is odd and $q=\frac{n-2}{2}$ if n is even.

$$f(c_{ji})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\}, j\equiv 1 \bmod 3,\ 1\le j\le q-1, i=1,2,\dots,r$$

$$f(c_{ji})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 2 \bmod 3,\ 2\le j\le q-1,\ i=1,2,\dots,r$$

$$f(c_{qi})=\frac{1}{3}\{(2n-2)i\},\ n \text{ is even or odd},\ i=1,2,\dots,r$$

$$f(c_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\}, j\equiv 1 \bmod 3, j\equiv 2 \bmod 3, 1\le j\le q-1$$

$$f(c_{ji}{}')=\frac{1}{3}\{(2n-2)i-(2n-(5+4j))\},\ j\equiv 0 \bmod 3,\ 3\le j\le q-1$$

$$f(c_{qi}{}')=\frac{1}{3}\{(2n-2)i\}-1,\ n \text{ is odd};\ f(c_{qi}{}')=\frac{1}{3}\{(2n-2)i\},\ n \text{ is even}$$

$$f(b_{ji}b_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-[2n-(5+4j)]\right\}, j\equiv 0 \bmod 3,\ 3\le j\le t-1,\ i=1,2,\dots,r$$

$f(a_i b_{1i}) = \frac{1}{3}\{(2n-2)i-(2n-5)\} f(b_{ti}a_{i+1}) = \left\lceil \frac{(2n-2)i+2}{3} \right\rceil$, n is even or odd, $i = 1, 2, \ldots, r$

$f(b_{ji}b_{ji}') = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(2+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}$, $j \equiv 1 \mod 3$; $j \equiv 2 \mod 3$; $1 \le j \le t-1$, $i = 1, 2, \ldots, r$

$f(b_{ji}b_{ji}') = \frac{1}{3}\{(2n-2)i - [2n-(2+4j)]\}$, $j \equiv 0 \mod 3$; $3 \le j \le t-1$, $i = 1, 2, \ldots, r$

$f(b_{ti}b_{ti}') = \frac{1}{3}\{(2n-2)i\}$($n$ is odd); $f(b_{ti}b_{ti}') = \frac{1}{3}\{(2n-2)i\} - 1$ (n is even), $i = 1, 2, \ldots, r$

$f(a_i c_{1i}) = \frac{1}{3}\{(2n-2)i-(2n-8)\}, i = 1, 2, \ldots, r; f(c_{qi}a_{i+1}) = \frac{(2n-2)i}{3}$ (n is odd or even)

$f(c_{ji}c_{(j+1)i}) = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(5+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}$, $j \equiv 1 \mod 3$; $j \equiv 2 \mod 3$; $1 \le j \le q-1$,

$f(c_{ji}c_{(j+1)i}) = \frac{1}{3}\{(2n-2)i - [2n-(8+4j)]\}$, $j \equiv 0 \mod 3$; $3 \le j \le q-1$, $i = 1, 2, \ldots, r$

$f(c_{qi}c_{qi}') = \frac{1}{3}\left\{(2n-2)i\right\} - 1$($n$ is odd); $f(c_{qi}c_{qi}') = \frac{1}{3}\{(2n-2)i\}$($n$ is even).

Case 2 : For $n \equiv 5 \mod 6$ and $n \ge 11$.

This case is divided into three subcases as follows.

Subcase 2.1 : For length $i \equiv 1 \mod 3$.

In this subcase, we provide labels of vertices and edges below.

$f(a_i) = \frac{1}{3}\{(2n-2)i-(2n-5)\}$; $f(a_{i+1}) = \left\lceil \frac{(2n-2)i+2}{3} \right\rceil$; $f(b_{1i}) = \frac{1}{3}\{(2n-2)i-(2n-5)\}$

$f(b_{ji}) = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(5+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}$, $j \equiv 1 \mod 3, j \equiv 2 \mod 3$, $2 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$

$f(b_{ji}) = \frac{1}{3}\left\{(2n-2)i - \left[2n-(2+4j)\right]\right\}$, $j \equiv 0 \mod 3$; $3 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$,

$f(b_{1i}') = \frac{1}{3}\left\{(2n-2)r - [2n-8]\right\}$,

$f(b_{ji}') = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(2+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}$, $j \equiv 1 \mod 3$; $4 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$,

$f(b_{ji}') = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(5+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}$, $j \equiv 2 \mod 3$,

$f(b_{ji}') = \frac{1}{3}\left\{(2n-2)i - [2n-(2+4j)]\right\}$, $j \equiv 0 \mod 3$,

$$f(c_{ji})=\frac{1}{3}\left\{(2n-2)i-[2n-(5+4j)]\right\},\ j\equiv 0 \mod 3;\ 3\le j\le \left\lfloor \frac{n-2}{2}\right\rfloor,$$

$$f(c_{ji})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 1 \mod 3;\ 1\le j\le \left\lfloor \frac{n-2}{2}\right\rfloor$$

$$f(c_{ji})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lceil \frac{j}{3}\right\rceil\right)\right]\right\},\ j\equiv 2 \mod 3;\ 2\le j\le \left\lfloor \frac{n-2}{2}\right\rfloor$$

$$f(c_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 1 \mod 3,\ j\equiv 2 \mod 3,\ 1\le j\le \left\lfloor \frac{n-2}{2}\right\rfloor$$

$$f(c_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-(2n-(5+4j))\right\},\ j\equiv 0 \mod 3;\ 3\le j\le \left\lfloor \frac{n-2}{2}\right\rfloor,$$

$$f(a_i b_{1i})=\frac{1}{3}\left\{(2n-2)i-(2n-5)\right\};\ f(b_{1i}b_{2i})=\frac{1}{3}\left\{(2n-2)i-(2n-11)\right\},$$

$$f(b_{ji}b_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 1 \mod 3;\ 4\le j\le \left\lceil \frac{n-2}{2}\right\rceil-1$$

$$f(b_{ji}b_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 2 \mod 3;\ 2\le j\le \left\lceil \frac{n-2}{2}\right\rceil-1$$

$$f(b_{ji}b_{(j+1)i})=\frac{1}{3}\{(2n-2)i-[2n-(5+4j)]\},\ j\equiv 0 \mod 3;\ 3\le j\le \left\lceil \frac{n-2}{2}\right\rceil-1$$

$$f\left(b_{\left(\left\lceil \frac{n-2}{2}\right\rceil\right)i}a_{i+1}\right)=\frac{(2n-2)i+1}{3};\ f\left(c_{\left(\left\lfloor \frac{n-2}{2}\right\rfloor\right)i}a_{i+1}\right)=\frac{(2n-2)i+1}{3},$$

$$f(b_{ji}b_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(2+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 1 \mod 3,\ j\equiv 2 \mod 3,\ 1\le j\le \left\lceil \frac{n-2}{2}\right\rceil,$$

$$f(b_{ji}b_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-[2n-(2+4j)]\right\},\ j\equiv 0 \mod 3;\ 3\le j\le \left\lceil \frac{n-2}{2}\right\rceil,$$

$$f(a_i c_{1i})=\frac{1}{3}\left\{(2n-2)i-(2n-8)\right\}$$

$$f(c_{ji}c_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lceil \frac{j}{3}\right\rceil\right)\right]\right\},\ j\equiv 1 \mod 3,\ j\equiv 2 \mod 3,\ 1\le j\le \left\lfloor \frac{n-2}{2}\right\rfloor-1$$

$$f(c_{ji}c_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-[2n-(8+4j)]\right\},\ j\equiv 0 \mod 3;\ 3\le j\le \left\lfloor \frac{n-2}{2}\right\rfloor-1$$

$$f(c_{ji}c_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 1 \mod 3,\ j\equiv 2 \mod 3,\ 1\le j\le \left\lfloor \frac{n-2}{2}\right\rfloor,$$

$$f(c_{ji}c_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-(2n-(2+4j))\right\},\ j\equiv 0 \mod 3;\ 3\le j\le \left\lfloor \frac{n-2}{2}\right\rfloor.$$

Subcase 2.2 : For length $i \equiv 2 \bmod 3$.

We construct labels of elements of $C_r(C_n^{n-2})$ as follows:

$$f(a_i) = \frac{1}{3}\{(2n-2)i - (2n-6)\};\ f(a_{i+1}) = \left\lceil \frac{(2n-2)i+2}{3} \right\rceil$$

$$f(b_{ji}) = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(2+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}, j \equiv 1 \bmod 3,\ 1 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$$

$$f(b_{ji}) = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(5+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\},\ j \equiv 2 \bmod 3,\ 2 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$$

$$f(b_{ji}) = \frac{1}{3}\{(2n-2)i - [2n - (2+4j)]\}, j \equiv 0 \bmod 3, 3 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$$

$$f(b_{ji}') = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(2+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\},\ j \equiv 1 \bmod 3;\ j \equiv 2 \bmod 3;\ 1 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$$

$$f(b_{ji}') = \frac{1}{3}\{(2n-2)i - [2n - (2+4j)]\}, j \equiv 0 \bmod 3; 3 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$$

$$f(c_{ji}) = \frac{1}{3}\{(2n-2)i - [2n - (5+4j)]\}, j \equiv 0 \bmod 3;\ 3 \le j \le \left\lfloor \frac{n-2}{2} \right\rfloor$$

$$f(c_{ji}) = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(5+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}, j \equiv 1 \bmod 3, j \equiv 2 \bmod 3, 1 \le j \le \left\lfloor \frac{n-2}{2} \right\rfloor$$

$$f(c_{ji}') = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(5+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}, j \equiv 1 \bmod 3, j \equiv 2 \bmod 3, 1 \le j \le \left\lfloor \frac{n-2}{2} \right\rfloor$$

$$f(c_{ji}') = \frac{1}{3}\{(2n-2)i - (2n - (2+4j))\}, j \equiv 0 \bmod 3; 3 \le j \le \left\lfloor \frac{n-2}{2} \right\rfloor$$

$$f(a_i b_{1i}) = \frac{1}{3}\{(2n-2)i - (2n-3)\}; f\left(b_{\left(\left\lceil \frac{n-2}{2} \right\rceil\right)i} a_{i+1}\right) = \left\lceil \frac{(2n-2)i+2}{3} \right\rceil$$

$$f(b_{ji} b_{(j+1)i}) = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(5+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}, j \equiv 1 \bmod 3;$$
$$j \equiv 2 \bmod 3, 1 \le j \le \left\lceil \frac{n-2}{2} \right\rceil - 1$$

$$f(b_{ji} b_{(j+1)i}) = \frac{1}{3}\{(2n-2)i - [2n - (5+4j)]\}, j \equiv 0 \bmod 3;\ 3 \le j \le \left\lceil \frac{n-2}{2} \right\rceil - 1$$

$$f(b_{ji} b_{ji}') = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(2+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}, j \equiv 2 \bmod 3, j \equiv 1 \bmod 3; 1 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$$

$$f(b_{ji} b_{ji}') = \frac{1}{3}\{(2n-2)i - [2n - (4j-1)]\}, j \equiv 0 \bmod 3;\ 3 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$$

$f(a_i c_{1i}) = \frac{1}{3}\{(2n-2)r-(2n-6)\}; f\left(c_{\left(\left\lceil\frac{n-2}{2}\right\rceil\right)i} a_{i+1}\right) = \left\lceil\frac{(2n-2)i+2}{3}\right\rceil$

$f(c_{ji}c_{(j+1)i}) = \frac{1}{3}\{(2n-2)i-[2n-(5+3j+3\lceil j/3\rceil)]\}, j \equiv 1 \bmod 3; j \equiv 2 \bmod 3,$
$1 \le j \le \lceil (n-2)/2\rceil - 1$

$f(c_{ji}c_{(j+1)i}) = \frac{1}{3}\{(2n-2)i-[2n-(5+4j)]\}, j \equiv 0 \bmod 3; 3 \le j \le \left\lfloor\frac{(n-2)}{2}\right\rfloor - 1$

$f(c_{ji}c_{ji}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(2+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\}, j \equiv 1 \bmod 3; 1 \le j < \left\lfloor\frac{n-2}{2}\right\rfloor$

$f(c_{ji}c_{ji}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\}, j \equiv 2 \bmod 3; 2 \le j < \left\lfloor\frac{n-2}{2}\right\rfloor$

$f(c_{ji}c_{ji}') = \frac{1}{3}\{(2n-2)i-(2n-(2+4j))\}, j \equiv 0 \bmod 3; 3 \le j < \left\lfloor\frac{n-2}{2}\right\rfloor$

Subcase 2.3 : For length $i \equiv 0 \bmod 3$.

We define labels of vertices and edges as shown below.

$f(a_i) = \frac{1}{3}\{(2n-2)i-(2n-4)\}; \; f(a_{i+1}) = \left\lceil\frac{(2n-2)i+2}{3}\right\rceil$

$f(b_{ji}) = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(4+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\}, j \equiv 1 \bmod 3; j \equiv 2 \bmod 3; 1 \le j \le \left\lceil\frac{n-2}{2}\right\rceil$

$f(b_{ji}) = \frac{1}{3}\{(2n-2)i-[2n-(4+4j)]\}, j \equiv 0 \bmod 3; 1 \le j \le \left\lceil\frac{n-2}{2}\right\rceil$

$f(b_{ji}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(4+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\}, j \equiv 1 \bmod 3; j \equiv 2 \bmod 3; 1 \le j \le \left\lceil\frac{n-2}{2}\right\rceil$

$f(b_{ji}') = \frac{1}{3}\{(2n-2)i-[2n-(1+4j)]\}, j \equiv 0 \bmod 3; 3 \le j \le \left\lceil\frac{n-2}{2}\right\rceil$

$f(c_{ji}) = \frac{1}{3}\{(2n-2)i-[2n-(4+4j)]\}, j \equiv 0 \bmod 3; 3 \le j \le \left\lfloor\frac{n-2}{2}\right\rfloor$

$f(c_{ji}) = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(4+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right]\right\}, j \equiv 1 \bmod 3; j \equiv 2 \bmod 3, 1 \le j \le \left\lfloor\frac{n-2}{2}\right\rfloor$

$f(c_{ji}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(4+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\}, j \equiv 1 \bmod 3; 1 \le j \le \left\lfloor\frac{n-2}{2}\right\rfloor$

$f(c_{ji}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(4+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right]\right\}, j \equiv 2 \bmod 3; 2 \le j \le \left\lfloor\frac{n-2}{2}\right\rfloor$

$f(c_{ji}') = \frac{1}{3}\{(2n-2)i-(2n-(4+4j))\}, j \equiv 0 \bmod 3; 3 \le j \le \left\lfloor\frac{n-2}{2}\right\rfloor$

$$f(b_{ji}b_{(j+1)i}) = \frac{1}{3}\left\{(2n-2)i-[2n-(4+4j)]\right\},\ j \equiv 0 \bmod 3;\ 3 \le j \le \left\lceil \frac{n-2}{2} \right\rceil - 1$$

$$f(b_{ji}b_{(j+1)i}) = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(4+3j+3\left\lceil \frac{j}{3} \right\rceil\right)\right]\right\},\ j \equiv 1 \bmod 3, j \equiv 2 \bmod 3.\ 1 \le j \le \left\lceil \frac{n-2}{2} \right\rceil - 1$$

$$f(a_i b_{1i}) = \frac{1}{3}\{(2n-2)i-(2n-4)\};\ f\left(b_{\left(\left\lceil \frac{(2n-2)i+2}{3} \right\rceil\right)_i} a_{i+1}\right) = \left\lceil \frac{(2n-2)i+2}{3} \right\rceil$$

$$f(b_{ji}b_{ji}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(1+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\},\ j \equiv 1 \bmod 3;\ 1 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$$

$$f(b_{ji}b_{ji}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(4+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\},\ j \equiv 2 \bmod 3;\ 2 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$$

$$f(b_{ji}b_{ji}') = \frac{1}{3}\{(2n-2)i-[2n-(1+4j)]\},\ j \equiv 0 \bmod 3;\ 3 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$$

$$f(a_i c_{1i}) = \frac{1}{3}\{(2n-2)i-(2n-7)\},\ f\left(c_{\left(\left\lfloor \frac{n-2}{2} \right\rfloor\right)_i} a_{i+1}\right) = \frac{(2n-2)i}{3}$$

$$f(c_{ji}c_{(j+1)i}) = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(4+3j+3\left\lceil \frac{j}{3} \right\rceil\right)\right]\right\},\ j \equiv 1 \bmod 3;\ 1 \le j < \left\lfloor \frac{n-2}{2} \right\rfloor - 1$$

$$f(c_{ji}c_{(j+1)i}) = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(7+3j+3\left\lceil \frac{j}{3} \right\rceil\right)\right]\right\},\ j \equiv 2 \bmod 3;\ 1 \le j < \left\lfloor \frac{n-2}{2} \right\rfloor - 1$$

$$f(c_{ji}c_{(j+1)i}) = \frac{1}{3}\{(2n-2)i-[2n-(7+4j)]\},\ j \equiv 0 \bmod 3;\ 3 \le j < \left\lfloor \frac{n-2}{2} \right\rfloor - 1$$

$$f(c_{ji}c_{ji}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(4+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\},\ j \equiv 1 \bmod 3;\ j \equiv 2 \bmod 3, 1 \le j \le \left\lfloor \frac{n-2}{2} \right\rfloor$$

$$f(c_{ji}c_{ji}') = \frac{1}{3}\left\{(2n-2)i-(2n-(4+4j))\right\},\ j \equiv 0 \bmod 3;\ 3 \le j < \left\lfloor \frac{n-2}{2} \right\rfloor$$

Case 3 : For $n \equiv 3 \bmod 6$ and $n \ge 9$.

We deal with three subcases.

Subcase 3.1 : For $n \equiv 3 \bmod 6$ and length $i \equiv 1 \bmod 3$.

In this subcase, we define labels for elements of $C_r(C_n^{n-2})$ as follows.

$$f(a_i) = \frac{1}{3}\{(2n-2)i-(2n-5)\};\ f(b_{1i}) = \frac{1}{3}\{(2n-2)i-(2n-5)\}$$

$$f(b_{ji}) = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\},\ j \equiv 1 \bmod 3;\ j \equiv 2 \bmod 3;\ 1 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$$

$$f(b_{ji}) = \frac{1}{3}\{(2n-2)i-[2n-(2+4j)]\},\ j \equiv 0 \bmod 3;\ 3 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$$

$f(a_{i+1})=\left\lceil \frac{(2n-2)i+2}{3}\right\rceil;\ f(b_{1i}{}')=\frac{1}{3}\left\{(2n-2)r-[2n-8]\right\}$

$f(b_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(2+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 1 \bmod 3;\ 4\le j\le \left\lceil \frac{n-2}{2}\right\rceil$

$f(b_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 2 \bmod 3,\ 2\le j\le \left\lceil \frac{n-2}{2}\right\rceil$

$f(b_{ji}{}')=\frac{1}{3}\{(2n-2)i-[2n-(2+4j)]\},\ j\equiv 0 \bmod 3;\ 3\le j\le \left\lceil \frac{n-2}{2}\right\rceil$

$f(c_{ji})=\frac{1}{3}\{(2n-2)i-[2n-(5+4j)]\},3\le j\le \left\lfloor \frac{n-2}{2}\right\rfloor,\ j\equiv 0 \bmod 3$

$f(c_{ji})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 1 \bmod 3;\ 1\le j\le \left\lfloor \frac{n-2}{2}\right\rfloor$

$f(c_{ji})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lceil \frac{j}{3}\right\rceil\right)\right]\right\},\ j\equiv 2 \bmod 3;\ 2\le j\le \left\lfloor \frac{n-2}{2}\right\rfloor$

$f(a_ib_{1i})=\frac{1}{3}\left\{(2n-2)i-(2n-5)\right\};\ f(b_{1i}b_{2i})=\frac{1}{3}\{(2n-2)i-(2n-11)\}$

$f(b_{ji}b_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 1 \bmod 3;\ 4\le j\le \left\lceil \frac{n-2}{2}\right\rceil-1$

$f(b_{ji}b_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lceil \frac{j}{3}\right\rceil\right)\right]\right\},\ j\equiv 2 \bmod 3;\ 2\le j\le \left\lceil \frac{n-2}{2}\right\rceil-1$

$f(b_{ji}b_{(j+1)i})=\frac{1}{3}\{(2n-2)i-[2n-(5+4j)]\},\ j\equiv 0 \bmod 3;\ 3\le j\le \left\lceil \frac{n-2}{2}\right\rceil-1$

$f(b_{ji}b_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(2+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 1 \bmod 3;\ j\equiv 2 \bmod 3;\ 1\le j\le \left\lceil \frac{n-2}{2}\right\rceil$

$f(b_{ji}b_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-[2n-(2+4j)]\right\},\ j\equiv 0 \bmod 3;\ 3\le j\le \left\lceil \frac{n-2}{2}\right\rceil$

$f\left(b_{\left(\left\lceil \frac{n-2}{2}\right\rceil\right)i}a_{i+1}\right)=\left\lceil \frac{(2n-2)i+2}{3}\right\rceil;\ f\left(c_{\left(\left\lfloor \frac{n-2}{2}\right\rfloor\right)i}a_{i+1}\right)=\left\lceil \frac{(2n-2)i+2}{3}\right\rceil;$

$f(a_ic_{1i})=\frac{1}{3}\left\{(2n-2)i-(2n-8)\right\}$

$f(c_{ji}c_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lceil \frac{j}{3}\right\rceil\right)\right]\right\},\ j\equiv 1 \bmod 3,\ j\equiv 2 \bmod 3.1\le j\le \left\lceil \frac{n-2}{2}\right\rceil-1$

$f(c_{ji}c_{(j+1)i})=\frac{1}{3}\{(2n-2)i-[2n-(8+4j)]\},\ j\equiv 0 \bmod 3;\ 3\le j< \left\lfloor \frac{n-2}{2}\right\rfloor-1$

$f(c_{ji}c_{ji}{}') = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(5+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}, j \equiv 1 \text{ mod } 3; j \equiv 2 \text{ mod } 3, 1 \le j \le \left\lfloor \frac{n-2}{2} \right\rfloor$

$f(c_{ji}c_{ji}{}') = \frac{1}{3}\{(2n-2)i - (2n-(2+4j))\},\ j \equiv 0 \text{ mod } 3;\ 3 \le j < \left\lfloor \frac{n-2}{2} \right\rfloor$

Subcase 3.2 : For $n \equiv 3 \text{mod} 6$ and length $i \equiv 2 \text{mod} 3$.

All elements of $C_r(C_n^{n-2})$ are labeled as in Subcase 2.3, except for labels of the following edges:

$$f\left(b_{\left\lceil \frac{n-2}{2} \right\rceil i} a_{i+1}\right) = \frac{(2n-2)i+1}{3} \text{ and } f\left(c_{\left\lfloor \frac{n-2}{2} \right\rfloor i} a_{i+1}\right) = \frac{(2n-2)i+1}{3}.$$

Subcase 3.3 : For $n \equiv 3 \text{ mod} 6$ and length $i \equiv 0 \text{ mod} 3$.

In this case, we assign labels for each $v, e \in V(C_r(C_n^{n-2})) \cup E(C_r(C_n^{n-2}))$ in the following way.

$f(a_i) = \frac{1}{3}\{(2n-2)i - (2n-6)\};\ f(a_{i+1}) = \left\lceil \frac{(2n-2)i+2}{3} \right\rceil$

$f(b_{ji}) = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(3+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}, j \equiv 1 \text{ mod } 3;\ 1 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$

$f(b_{ji}) = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(6+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}, j \equiv 2 \text{ mod } 3;\ 2 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$

$f(b_{ji}) = \frac{1}{3}\{(2n-2)i - [2n-(3+4j)]\}, j \equiv 0 \text{ mod } 3;\ 3 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$

$f(b_{ji}{}') = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(3+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}, j \equiv 1 \text{ mod } 3;\ j \equiv 2 \text{ mod } 3;\ 1 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$

$f(b_{ji}{}') = \frac{1}{3}\{(2n-2)i - [2n-(3+4j)]\}, j \equiv 0 \text{ mod } 3;\ 3 \le j \le \left\lceil \frac{n-2}{2} \right\rceil$

$f(c_{ji}) = \frac{1}{3}\{(2n-2)i - [2n-(6+4j)]\},\ j \equiv 0 \text{ mod } 3;\ 3 \le j \le \left\lfloor \frac{n-2}{2} \right\rfloor$

$f(c_{ji}) = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(6+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\},\ j \equiv 1 \text{ mod } 3;\ j \equiv 2 \text{ mod } 3,\ 1 \le j \le \left\lfloor \frac{n-2}{2} \right\rfloor$

$f(c_{ji}{}') = \frac{1}{3}\left\{(2n-2)i - \left[2n - \left(6+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\},\ j \equiv 1 \text{ mod } 3;\ j \equiv 2 \text{ mod } 3,\ 1 \le j \le \left\lfloor \frac{n-2}{2} \right\rfloor$

$f(c_{ji}{}') = \frac{1}{3}\{(2n-2)i - (2n-(3+4j))\},\ j \equiv 0 \text{ mod } 3;\ 3 \le j \le \left\lfloor \frac{n-2}{2} \right\rfloor$

$f(a_i b_{1i}) = \frac{1}{3}\{(2n-2)i - (2n-3)\};$

$$f\left(b_{\left(\left\lceil\frac{n-2}{2}\right\rceil\right)_i}a_{i+1}\right)=\left\lceil\frac{(2n-2)i+2}{3}\right\rceil;\ f\left(c_{\left(\left\lfloor\frac{n-2}{2}\right\rfloor\right)_i}a_{i+1}\right)=\frac{(2n-2)i}{3}$$

$$f(b_{ji}b_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(6+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 1 \bmod 3;\ j\equiv 2 \bmod 3,\ 1\le j\le\left\lceil\frac{n-2}{2}\right\rceil-1$$

$$f(b_{ji}b_{(j+1)i})=\frac{1}{3}\{(2n-2)i-[2n-(6+4j)]\},\ j\equiv 0 \bmod 3;\ 3\le j\le\left\lceil\frac{n-2}{2}\right\rceil-1$$

$$f(b_{ji}b_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(3+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 1 \bmod 3;\ j\equiv 2 \bmod 3;\ 1\le j\le\left\lceil\frac{n-2}{2}\right\rceil$$

$$f(b_{ji}b_{ji}{}')=\frac{1}{3}\{(2n-2)i-[2n-4j]\},\ j\equiv 0 \bmod 3;\ 3\le j\le\left\lceil\frac{n-2}{2}\right\rceil;$$

$$f(a_ic_{1i})=\frac{1}{3}\{(2n-2)r-(2n-6)\}$$

$$f(c_{ji}c_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(6+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right]\right\},\ j\equiv 1 \bmod 3;\ j\equiv 2 \bmod 3,\ 1\le j\le\left\lfloor\frac{n-2}{2}\right\rfloor-1$$

$$f(c_{ji}c_{(j+1)i})=\frac{1}{3}\{(2n-2)i-[2n-(6+4j)]\},\ j\equiv 0 \bmod 3;\ 3\le j<\left\lfloor\frac{n-2}{2}\right\rfloor-1$$

$$f(c_{ji}c_{ji}{}')=\frac{1}{3}\{(2n-2)i-(2n-(3+4j))\},\ j\equiv 0 \bmod 3;\ 3\le j\le\left\lfloor\frac{n-2}{2}\right\rfloor$$

$$f(c_{ji}c_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(3+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 1 \bmod 3;\ 1\le j\le\left\lfloor\frac{n-2}{2}\right\rfloor$$

$$f(c_{ji}c_{ji}{}')=\frac{1}{3}\left\{(2n-2)i-\left[2n-\left(6+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\},\ j\equiv 2 \bmod 3;\ 1\le j\le\left\lfloor\frac{n-2}{2}\right\rfloor$$

In Case 1, 2, and 3 (for all subcases), it is shown that labels of all elements of $C_r(C_n^{n-2})$ are not more than $k=\left\lceil\frac{(2n-2)r+2}{3}\right\rceil$. Further, we show that the weights $wt(e)\ne wt(e')$ whenever $e\ne e'$:

$$wt(a_ib_{1i})=(2n-2)i-\{2n-5\},wt\left(b_{\left(\left\lceil\frac{n-2}{2}\right\rceil\right)_i}a_{(i+1)}\right)=(2n-2)i+2,\ wt\left(c_{\left(\left\lfloor\frac{n-2}{2}\right\rfloor\right)_i}a_{(i+1)}\right)$$
$$=(2n-2)i+1,\ wt(b_{ji}b_{(j+1)i})=(2n-2)i-\{2n-(5+4j)\},1\le j\le\left\lceil\frac{n-2}{2}\right\rceil-1;$$
$$wt(a_ic_{1i})=(2n-2)i-\{2n-7\},wt(b_{ji}b_{ji}{}')=(2n-2)i-\{2n-(2+4j)\};$$
$$wt(c_{ji}c_{ji}{}')=(2n-2)i-\{2n-(4+4j)\},1\le j\le\left\lfloor\frac{n-2}{2}\right\rfloor wt(c_{ji}c_{(j+1)i}=$$

$(2n-2)i-\{2n-(7+4j)\}, 1 \le j \le \left\lfloor \frac{n-2}{2} \right\rfloor - 1;$

It is obvious that f is an edge irregular total k-labeling on the general cactus chain graphs. This concludes $tes(C_r(C_n^{n-2})) = k = \left\lceil \frac{((2n-2)r+2)}{3} \right\rceil$.

Case 4 : For $n \equiv 0 \bmod 6$, $n \equiv 2 \bmod 6$, and $n \ge 6$.

This case is divided into three subcases as follows.

Subcase 4.1 : For length $i \equiv 1 \bmod 3$.

In this case, labels for $x, e \in V(C_r(C_n^{n-2})) \cup E(C_r(C_n^{n-2}))$ are given below.

$f(a_i) = \frac{1}{3}\{(2n-2)i-(2n-5)\}$; $f(a_i) = \left\lceil \frac{(2n-2)i+2}{3} \right\rceil$; $f(b_{1i}) = \frac{1}{3}\{(2n-2)i-(2n-5)\}$;

$$f(b_{ji}) = \begin{cases} \frac{1}{3}\{(2n-2)i-[2n-(5+3j)]\}, & j = 2,3 \\ \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(2+3j+3\left\lceil \frac{j}{3} \right\rceil\right)\right]\right\} & 4 \le j \le \frac{n-2}{2} \end{cases}$$

$$f(b_{ji}') = \begin{cases} \frac{1}{3}\{(2n-2)i-(2n-8)\} & j = 1,2, \\ \frac{1}{3}\{(2n-2)i-(2n-(5+3j))\} & j = 3,4, \\ \frac{1}{3}\{(2n-2)i-(2n-(8+3j))\} & j = 5, \end{cases}$$

$$f(b_{ji}') = \frac{1}{3}\{(2n-2)i-(2n-(2+4j)\}, j = 0 \bmod 3, \ 6 \le j \le \frac{n-2}{2}$$

$$f(b_{ji}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(2+3j+3\left\lceil \frac{j}{3} \right\rceil\right)\right]\right\}, j = 2 \bmod 3, \ 8 \le j \le \frac{n-2}{2}$$

$$f(b_{ji}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(2+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}, j = 1 \bmod 3, \ 7 \le j \le \frac{n-2}{2}$$

$$f(c_{ji}') = \frac{1}{3}\{(2n-2)i-[2n-(5+3j)]\}, \ j = 1,2;$$

$$f(c_{ji}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right]\right\}, 3 \le j \le \frac{n-2}{2}$$

$$f(c_{ji}) = \begin{cases} \frac{1}{3}\{(2n-2)i-(2n-8)\} & j=1 \\ \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lceil \frac{j}{3}\right\rceil\right)\right]\right\} & j \equiv 2 \ \mathrm{mod} 3,\ 2 \le j \le \frac{n-2}{2} \\ \frac{1}{3}\{(2n-2)i-[2n-(5+4j)]\} & j \equiv 0 \ \mathrm{mod} 3,\ 3 \le j \le \frac{n-2}{2} \\ \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\} & j \equiv 1 \ \mathrm{mod} 3,\ 4 \le j \le \frac{n-2}{2} \end{cases}$$

$$f\left(b_{\frac{n-2}{2}i}a_{i+1}\right) = \left\lceil \frac{(2n-2)i+2}{3} \right\rceil - 1;\ f\left(c_{\frac{n-2}{2}i}a_{i+1}\right) = \left\lceil \frac{(2n-2)i+2}{3} \right\rceil - 1;$$

$$f(a_i b_{1i}) = \frac{1}{3}\{(2n-2)i-(2n-5)\}$$

$$f(a_i c_{1i}) = \frac{1}{3}\{(2n-2)i-(2n-8)\};\ f(b_{1i}b_{1i}{}') = \frac{1}{3}\{(2n-2)i-(2n-5)\}$$

$$f(c_{ji}c_{(j+1)i}) = \begin{cases} \frac{1}{3}\{(2n-2)i-(2n-(8+3j))\} & j=1,2 \\ \frac{1}{3}\{(2n-2)i-(2n-(8+4j))\} & j=0 \ \mathrm{mod}\ 3,\ 3 \le j \le \frac{n-2}{2}-1 \\ \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lceil \frac{j}{3}\right\rceil\right)\right]\right\} & j=1 \ \mathrm{mod}\ 3, j=2 \ \mathrm{mod}\ 3, \\ & \quad 4 \le j \le \frac{n-2}{2}-1 \end{cases}$$

$$f(c_{ji}c_{ji}{}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\}, j=1 \ \mathrm{mod}\ 3,\ j=2 \ \mathrm{mod}\ 3, 1 \le j \le \frac{n-2}{2}$$

$$f(c_{ji}c_{ji}{}') = \frac{1}{3}\{(2n-2)i-(2n-(2+4j))\}, j=0 \ \mathrm{mod}\ 3, 3 \le j \le \frac{n-2}{2}$$

$$b_{ji}b_{(j+1)i} = \begin{cases} \frac{1}{3}\{(2n-2)i-(2n-(8+3j))\}, & j=1,2,3,4 \\ \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lceil \frac{j}{3}\right\rceil\right)\right]\right\}, & j \equiv 2 \ \mathrm{mod}\ 3, j \ge 5 \\ \frac{1}{3}\{(2n-2)i-(2n-(5+4j))\}, & j \equiv 0 \ \mathrm{mod}\ 3, j \ge 6 \\ \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(5+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\}, & j \equiv 1 \ \mathrm{mod}\ 3, j \ge 7 \end{cases}$$

$$f(b_{ji}b_{ji}{}') = \begin{cases} \frac{1}{3}\{(2n-2)i-(2n-(5+3j))\} & j=2,3, \\ \frac{1}{3}\{(2n-2)i-(2n-(8+3j))\} & j=4,5 \\ \frac{1}{3}\{(2n-2)i-(2n-(2+4j))\} & j=0 \ \mathrm{mod}\ 3, 6 \le j \le \frac{n-2}{2} \\ \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(2+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right]\right\} & j=1 \ \mathrm{mod}\ 3, j=2 \ \mathrm{mod}\ 3, 7 \le j \le \frac{n-2}{2} \end{cases}$$

Subcase 4.2 : For length $i \equiv 2 \bmod 3$.

We create labels for elements $x \in V(C_r(C_n^{n-2}))$ and $e \in E(C_r(C_n^{n-2}))$ as presented below.

$$f(a_i) = \begin{cases} \frac{1}{3}\{(2n-2)i-(2n-4)\}, & n \equiv 0 \bmod 6 \\ \frac{1}{3}\{(2n-2)i-(2n-6)\}, & n \equiv 2 \bmod 6 \end{cases}$$

$$f(b_{ji}) = \begin{cases} \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(4+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right]\right\}, & 1 \le j \le \frac{n-2}{2},\ n = 0 \bmod 6 \\ \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(3+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right]\right\}, & 1 \le j \le \frac{n-2}{2},\ n = 2 \bmod 6 \end{cases}$$

$$f(b_{ji}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(3+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right]\right\}, 1 \le j \le \frac{n-2}{2}, n = 2 \bmod 6;$$

$$f(b_{ji}') = \begin{cases} \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(1+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right]\right\}, & j \equiv 1 \bmod 3, n \equiv 0 \bmod 6 \\ \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(4+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right]\right\}, & j \equiv 0 \bmod 3, j \equiv 2 \bmod 3, n \equiv 0 \bmod 6 \end{cases}$$

$$f(c_{ji}') = \frac{1}{3}\{(2n-2)i-(2n-9)\}, j = 1, 2;$$

$$f(c_{ji}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(3+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\}, 3 \le j \le \frac{n-2}{2}; n \equiv 2 \bmod 6$$

$$f(c_{ji}') = \frac{1}{3}\left\{(2n-2)i-\left[2n-\left(4+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right]\right\}, 1 \le j \le \frac{n-2}{2}, n = 0 \bmod 6$$

$$f(c_{ji}) = \begin{cases} \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(1+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right)\right\}, & j = 2 \bmod 3, n \equiv 0 \bmod 6 \\ \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(3+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right)\right\}, & j = 2 \bmod 3, n \equiv 2 \bmod 6 \end{cases}$$

$$f(c_{ji}) = \begin{cases} \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(4+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right)\right\}, & j = 1 \bmod 3, n \equiv 0 \bmod 6 \\ \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(3+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right)\right\}, & j = 1 \bmod 3, n \equiv 2 \bmod 6 \end{cases}$$

$$f(c_{ji}) = \begin{cases} \frac{1}{3}\{(2n-2)i-(2n-(4+4j))\}, & j = 0 \bmod 3, n \equiv 0 \bmod 6 \\ \frac{1}{3}\{(2n-2)i-(2n-(3+4j))\}, & j = 0 \bmod 3, n \equiv 2 \bmod 6 \end{cases}$$

$$f(a_i b_{1i}) = \tfrac{1}{3}\{(2n-2)i-(2n-7)\},\ n \equiv 0 \mod 6;$$

$$f(a_i b_{1i}) = \tfrac{1}{3}\{(2n-2)i-(2n-6)\}, n \equiv 2 \mod 6$$

$$f(b_{ji} b_{(j+1)i}) = \begin{cases} \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(7+3j+3\left\lceil \frac{j}{3}\right\rceil\right)\right)\right\} & j=2 \mod 3, n=0 \mod 6 \\ \frac{1}{3}\{(2n-2)i-(2n-(7+4j))\} & j=0 \mod 3, n=0 \mod 6 \\ \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(4+3j+3\left\lceil \frac{j}{3}\right\rceil\right)\right)\right\} & j=1 \mod 3, n=0 \mod 6 \end{cases}$$

$$f(b_{ji} b_{(j+1)i}) = \begin{cases} \frac{1}{3}\{(2n-2)i-(2n-(6+4j))\} & j \equiv 0 \mod 3; n \equiv 2 \mod 6 \\ \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(6+3j+3\left\lceil \frac{j}{3}\right\rceil\right)\right)\right\} & j \equiv 1 \mod 3, j \equiv 2 \mod 3; \\ & n \equiv 2 \mod 6 \end{cases}$$

$$f\left(b_{\frac{n-2}{2}i} a_{i+1}\right) = \left\lfloor \frac{(2n-2)i+2}{3} \right\rfloor;\ f\left(c_{\frac{n-2}{2}i} a_{i+1}\right) = \left\lfloor \frac{(2n-2)i+2}{3} \right\rfloor$$

$$f(b_{ji} b_{ji}') = \begin{cases} \frac{1}{3}\{(2n-2)i-(2n-(4+4j))\} & j \equiv 0 \mod 3;\ n \equiv 0 \mod 6 \\ \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(4+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right)\right\} & j \equiv 1 \mod 3, j \equiv 2 \mod 3; n \equiv 0 \mod 6 \end{cases}$$

$$f(b_{ji} b_{ji}') = \begin{cases} \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(6+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right)\right\} & j \equiv 2 \mod 3, n \equiv 2 \mod 6 \\ \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(3+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right)\right\} & j \equiv 1 \mod 3, n \equiv 2 \mod 6 \\ \frac{1}{3}\{(2n-2)i-(2n-(3+4j))\} & j \equiv 0 \mod 3, n \equiv 2 \mod 6 \end{cases}$$

$$f(a_i c_{1i}) = \tfrac{1}{3}\{(2n-2)i-(2n-4)\},\ n \equiv 0 \mod 6;$$

$$f(a_i c_{1i}) = \tfrac{1}{3}\{(2n-2)i-(2n-3)\},\ n \equiv 2 \mod 6$$

$$f(c_{ji} c_{(j+1)i}) = \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(6+3j+3\left\lceil \frac{j}{3}\right\rceil\right)\right)\right\},\ j \equiv 1 \mod 3, j \equiv 2 \mod 3$$
$$1 \le j < \left(\frac{n-2}{2}-1\right); n \equiv 0 \mod 6$$

$$f(c_{ji} c_{(j+1)i}) = \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(6+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right)\right\},\ j = \frac{n-2}{2}-1;\ n \equiv 0 \mod 6$$

$$f(c_{ji} c_{(j+1)i}) = \tfrac{1}{3}\{(2n-2)i-(2n-(6+4j))\}, j \equiv 0 \mod 3;\ n \equiv 0 \mod 6$$

$$f(c_{ji}c_{(j+1)i}) = \frac{1}{3}\left\{(2n-2)i - \left(2n - \left(4+3j+3\left\lceil \frac{j}{3} \right\rceil\right)\right)\right\}, j \equiv 1 \bmod 3;\ n \equiv 2 \bmod 6$$

$$f(c_{ji}c_{(j+1)i}) = \frac{1}{3}\left\{(2n-2)i - \left(2n - \left(7+3j+3\left\lceil \frac{j}{3} \right\rceil\right)\right)\right\}, j \equiv 2 \bmod 3,\ 2 \le j < \left(\frac{n-2}{2} - 1\right)\ n \equiv 2 \bmod 6$$

$$f(c_{ji}c_{(j+1)i}) = \frac{1}{3}\left\{(2n-2)i - \left(2n - \left(4+3j+3\left\lceil \frac{j}{3} \right\rceil\right)\right)\right\}, j = \frac{n-2}{2} - 1;\ n \equiv 2 \bmod 6$$

$$f(c_{ji}c_{(j+1)i}) = \frac{1}{3}\{(2n-2)i - (2n - (7+4j))\},\ j \equiv 0 \bmod 3;\ n \equiv 2 \bmod 6$$

$$f(c_{1i}c_{1i}') = \frac{1}{3}\{(2n-2)i - (2n-3)\},\ n \equiv 2 \bmod 6;$$

$$f(c_{1i}c_{1i}') = \frac{1}{3}\{(2n-2)i - (2n-4)\},\ n \equiv 0 \bmod 6$$

$$f(c_{ji}c_{ji}') = \frac{1}{3}\left\{(2n-2)i - \left(2n - \left(3+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right)\right\}, j \equiv 1 \bmod 3, j \equiv 2 \bmod 3; n \equiv 2 \bmod 6$$

$$f(c_{ji}c_{ji}') = \frac{1}{3}\{(2n-2)i - (2n-4j)\},\ j \equiv 0 \bmod 3, n \equiv 2 \bmod 6$$

$$f(c_{ji}c_{ji}') = \frac{1}{3}\left\{(2n-2)i - \left(2n - \left(4+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right)\right\}, j \equiv 2 \bmod 3;\ n \equiv 0 \bmod 6$$

$$f(c_{ji}c_{ji}') = \begin{cases} \frac{1}{3}\left\{(2n-2)i - \left(2n - \left(1+3j+3\left\lfloor \frac{j}{3} \right\rfloor\right)\right)\right\} & j \equiv 1 \bmod 3;\ n \equiv 0 \bmod 6 \\ \frac{1}{3}\{(2n-2)i - (2n - (1+4j))\} & j \equiv 0 \bmod 3;\ n \equiv 0 \bmod 6 \end{cases}$$

In Subcase 4.2 ($i \equiv 2 \bmod 3$), we verify the edge weights as follows:

$$wt(a_i b_{1i}) = (2n-2)i - \{2n-7\}, wt\left(b_{\left(\frac{n-2}{2}\right)i} a_{(i+1)}\right) = (2n-2)i + 2,$$

$$wt(b_{ji}b_{(j+1)i}) = (2n-2)i - \{2n - (7+4j)\},\ 1 \le j \le \frac{n-2}{2} - 1$$

$$wt(b_{ji}b_{ji}') = (2n-2)i - \{2n - (4+4j)\}, 1 \le j \le \frac{n-2}{2}$$

$$wt(c_{ji}c_{(j+1)i} = (2n-2)i - \{2n - (5+4j)\}, 1 \le j \le \frac{n-2}{2} - 1;$$

$$wt(a_i c_{1i}) = (2n-2)i - \{2n-5\},$$

$$wt(c_{ji}c_{ji}') = (2n-2)i - \{2n - (4+4j)\}, 1 \le j \le \frac{n-2}{2};\ wt\left(c_{\left(\frac{n-2}{2}\right)i} a_{(i+1)}\right) = (2n-2)i + 1,$$

Subcase 4.3 : For length $i \equiv 0 \bmod 3$.

We assign labels for elements $v, e \in V(C_r(C_n^{n-2})) \cup E(C_r(C_n^{n-2}))$ as displayed below.

$$f(a_i)=\begin{cases}\frac{1}{3}\{(2n-2)i-(2n-4)\} & n\equiv 2 \bmod 6\\ \frac{1}{3}\{(2n-2)i-(2n-6)\} & n\equiv 0 \bmod 6\end{cases}$$

$$f(b_{ji})=\begin{cases}\frac{1}{3}\{(2n-2)i-(2n-6j)\} & j=1,2;\ n\equiv 0 \bmod 6\\ \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(3+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right)\right\} & 3\le j<\frac{n-2}{2};\ n\equiv 0 \bmod 6\end{cases}$$

$$f(b_{ji})=\frac{1}{3}\left\{(2n-2)i-\left(2n-\left(3+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right)\right\}, 1\le j<\frac{n-2}{2};\ n\equiv 2 \bmod 6$$

$$f(b_{ji})=\left\lceil\frac{(2n-2)i+2}{3}\right\rceil,\ j=\frac{n-2}{2};\ n\equiv 2 \bmod 6$$

$$f(c_{1i}')=\frac{1}{3}\{(2n-2)i-(2n-7)\}, n\equiv 2 \bmod 6;\ f(c_{1i}')=\frac{1}{3}\{(2n-2)i-(2n-9)\},$$
$$n\equiv 0 \bmod 6$$

$$f(c_{ji}')=\frac{1}{3}\left\{(2n-2)i-\left(2n-\left(4+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right)\right\}, j\equiv 2 \bmod 3;\ 2\le j<\frac{n-2}{2}-1,\ n\equiv 2 \bmod 6$$

$$f(c_{ji}')=\frac{1}{3}\{(2n-2)i-(2n-(4+4j))\},\ j\equiv 0 \bmod 3;\ 3\le j<\frac{n-2}{2}-1,\ n\equiv 2 \bmod 6$$

$$f(c_{ji}')=\frac{1}{3}\left\{(2n-2)i-\left(2n-\left(4+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right)\right\},\ j\equiv 1 \bmod 3;\ 4\le j<\frac{n-2}{2}-1,\ n\equiv 2 \bmod 6$$

$$f(c_{ji}')=\frac{1}{3}\left\{(2n-2)i-\left(2n-\left(4+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right)\right\},\ j=\frac{n-2}{2}-1,\frac{n-2}{2};\ n\equiv 2 \bmod 6$$

$$f(c_{ji}')=\begin{cases}\frac{1}{3}\left\{(2n-2)i-\left(2n-\left(3+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right)\right\} & j\equiv 1 \bmod 3,\ j\equiv 2 \bmod 3;\ n\equiv 0 \bmod 6\\ \frac{1}{3}\{(2n-2)i-(2n-(3+4j))\} & j\equiv 0 \bmod 3;\ n\equiv 0 \bmod 6\end{cases}$$

$$f(c_{ji})=\frac{1}{3}\left\{(2n-2)i-\left(2n-\left(4+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right)\right\}, 1\le j<\frac{n-2}{2}, n\equiv 2 \bmod 6$$

$$f(c_{ji})=\begin{cases}\frac{1}{3}\left\{(2n-2)i-\left(2n-\left(3+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right)\right\} & j\equiv 1 \bmod 3,\ j\equiv 2 \bmod 3;\ n\equiv 0 \bmod 6\\ \frac{1}{3}\{(2n-2)i-(2n-(6+4j))\} & j\equiv 0 \bmod 3;\ n\equiv 0 \bmod 6\end{cases}$$

$$f(c_{ji})=\frac{(2n-2)i}{3},\ j=\frac{n-2}{2},\ n\equiv 0 \mod 6;\ n\equiv 2 \mod 6$$

$$f(a_i b_{1i})=\begin{cases}\frac{1}{3}\{(2n-2)i-(2n-3)\} & n\equiv 0 \mod 6\\ \frac{1}{3}\{(2n-2)i-(2n-4)\} & n\equiv 2 \mod 6\end{cases}$$

$$f(a_i c_{1i})=\begin{cases}\frac{1}{3}\{(2n-2)i-(2n-6)\} & n\equiv 0 \mod 6\\ \frac{1}{3}\{(2n-2)i-(2n-7)\} & n\equiv 2 \mod 6\end{cases}$$

$$f(b_{ji}b_{(j+1)i})=\begin{cases}\frac{1}{3}\left\{(2n-2)i-\left(2n-\left(6+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right)\right\} & j\equiv 1 \mod 3,\ j\equiv 2 \mod 3;\ n\equiv 0 \mod 6\\ \frac{1}{3}\{(2n-2)i-(2n-(6+4j))\} & j\equiv 0 \mod 3;\ n\equiv 0 \mod 6\end{cases}$$

$$f(b_{ji}b_{(j+1)i})=\begin{cases}\frac{1}{3}\left\{(2n-2)i-\left(2n-\left(4+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right)\right\} & j\equiv 1 \mod 3,\ j\equiv 2 \mod 3;\ n\equiv 2 \mod 6\\ \frac{1}{3}\{(2n-2)i-(2n-(4+4j))\} & j\equiv 0 \mod 3;\ n\equiv 2 \mod 6\end{cases}$$

$$f\left(b_{\frac{n-2}{2}i}a_{i+1}\right)=\frac{(2n-2)i}{3},\ f\left(c_{\frac{n-2}{2}i}a_{i+1}\right)=\frac{(2n-2)i}{3};\ n\equiv 0 \mod 6;\ n\equiv 2 \mod 6$$

$$f(c_{ji}c_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-\left(2n-\left(6+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right)\right\},\ j\equiv 1 \mod 3;\ j\equiv 2 \mod 3;$$
$$1\le j<\left(\frac{n-2}{2}-1\right);\ n\equiv 0 \mod 6$$

$$f(c_{ji}c_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-\left(2n-\left(6+3j+3\left\lfloor\frac{j}{3}\right\rfloor\right)\right)\right\},\ j=\frac{n-2}{2}-1;\ n\equiv 0 \mod 6$$

$$f(c_{ji}c_{(j+1)i})=\frac{1}{3}\{(2n-2)i-(2n-(6+4j))\},\ j\equiv 0 \mod 3;\ n\equiv 0 \mod 6$$

$$f(c_{ji}c_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-\left(2n-\left(4+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right)\right\},\ j\equiv 1 \mod 3;\ n\equiv 2 \mod 6$$

$$f(c_{ji}c_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-\left(2n-\left(7+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right)\right\},\ j\equiv 2 \mod 3;\ n\equiv 2 \mod 6,$$
$$2\le j<\left(\frac{n-2}{2}-1\right)$$

$$f(c_{ji}c_{(j+1)i})=\frac{1}{3}\left\{(2n-2)i-\left(2n-\left(4+3j+3\left\lceil\frac{j}{3}\right\rceil\right)\right)\right\},\ j=\left(\frac{n-2}{2}-1\right),\ n\equiv 2 \mod 6$$

$$f(c_{ji}c_{(j+1)i})=\frac{1}{3}\{(2n-2)i-(2n-(7+4j))\},\ j\equiv 0 \mod 6,\ n\equiv 2 \mod 6$$

$f(b_{1i}b_{1i}{}') = \frac{1}{3}\{(2n-2)i-(2n-3)\}, n \equiv 0 \mod 6;$

$f(b_{1i}b_{1i}{}') = \frac{1}{3}\{(2n-2)i-(2n-4)\}, n \equiv 2 \mod 6$

$f(b_{ji}b_{ji}{}') = \frac{1}{3}\{(2n-2)i-(2n-(3+3j))\},\ j = 2, 3,\ n \equiv 0 \mod 6$

$f(b_{ji}b_{ji}{}') = \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(3+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right)\right\}, j \equiv 1 \mod 3, j \equiv 2 \mod 3;$

$4 \le j \le \left(\frac{n-2}{2}\right),\ n \equiv 0 \mod 6$

$f(b_{ji}b_{ji}{}') = \frac{1}{3}\{(2n-2)i-(2n-4j)\},\ j \equiv 0 \mod 3, n \equiv 0 \mod 6$

$f(b_{ji}b_{ji}{}') = \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(4+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right)\right\},\ j \equiv 2 \mod 3, n \equiv 2 \mod 6$

$f(b_{ji}b_{ji}{}') = \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(1+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right)\right\},\ j \equiv 1 \mod 3, n \equiv 2 \mod 6$

$f(b_{ji}b_{ji}{}') = \frac{1}{3}\{(2n-2)i-(2n-(1+4j))\}, j \equiv 0 \mod 3, n \equiv 2 \mod 6$

$f(c_{ji}c_{ji}{}') = \frac{1}{3}\{(2n-2)i-(2n-(4+3j))\}, j = 1, 2; n \equiv 2 \mod 6$

$f(c_{ji}c_{ji}{}') = \frac{1}{3}\{(2n-2)i-(2n-(4+4j))\}, j \equiv 0 \mod 3,\ 3 \le j < \frac{n-2}{2}, n \equiv 2 \mod 6$

$f(c_{ji}c_{ji}{}') = \frac{1}{3}\{(2n-2)i-(2n-(1+4j))\}, j = \frac{n-2}{2}, n \equiv 2 \mod 6$

$f(c_{ji}c_{ji}{}') = \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(4+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right)\right\}, j \equiv 1 \mod 3, j \equiv 2 \mod 3, n \equiv 2 \mod 6$

$$f(c_{ji}c_{ji}{}') = \begin{cases} \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(6+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right)\right\} & j \equiv 2 \mod 3, n \equiv 0 \mod 6 \\ \frac{1}{3}\left\{(2n-2)i-\left(2n-\left(3+3j+3\left\lfloor \frac{j}{3}\right\rfloor\right)\right)\right\} & j \equiv 1 \mod 3, n \equiv 0 \mod 6 \\ \frac{1}{3}\{(2n-2)i-(2n-(3+4j))\} & j \equiv 0 \mod 3, n \equiv 0 \mod 6 \end{cases}$$

In Subcases 4.1 and 4.3 ($i \equiv 1 \mod 3$ and $i \equiv 0 \mod 3$), we observe the weights of edges below:

$wt(a_i b_{1i}) = (2n-2)i - \{2n-5\}, wt\left(b_{\left(\frac{n-2}{2}\right)i} a_{(i+1)}\right) = (2n-2)i+1;$

$wt(a_i c_{1i}) = (2n-2)i - \{2n-7\},$

$$wt(b_{ji}b_{(j+1)i}) = (2n-2)i - \{2n-(5+4j)\},$$
$$wt(c_{ji}c_{(j+1)i}) = (2n-2)i - \{2n-(7+4j)\}, 1 \le j \le \frac{n-2}{2} - 1$$
$$wt(b_{ji}b_{ji}') = (2n-2)i - \{2n-(2+4j)\}, 1 \le j \le \frac{n-2}{2};$$
$$wt\left(c_{\left(\frac{n-2}{2}\right)i}a_{(i+1)}\right) = (2n-2)i + 2,$$
$$wt(c_{ji}c_{ji}') = (2n-2)i - \{2n-(4+4j)\}, 1 \le j \le \frac{n-2}{2}.$$

In Case 4 (all subcases), no edges have a same weight. In addition, the vertex and edge labels are not more than $k = \left\lceil \frac{((2n-2)r+2)}{3} \right\rceil$. Thus, $tes(C_r(C_n^{n-2})) = \left\lceil \frac{(2n-2)r+2}{3} \right\rceil$. □

Example 2.2.1 : Figure 1 depicts a pattern to get $tes(C_4(C_{13}^{11})) = \left\lceil \frac{96+2}{3} \right\rceil = 33$.

Further, Figure 2 shows a pattern to get $tes(C_6(C_{11}^{9})) = \left\lceil \frac{120+2}{3} \right\rceil = 41$.

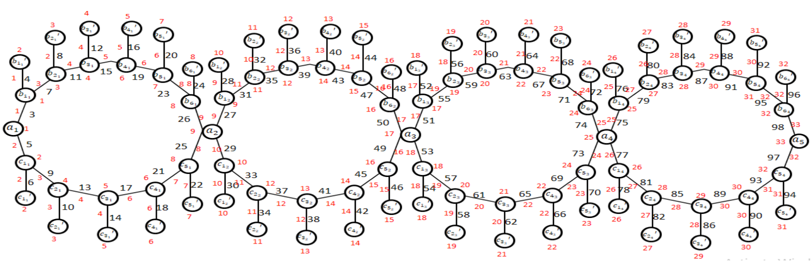

Fig. 1

An edge irregular total 33-labeling of $C_4(C_{13}^{11})$.

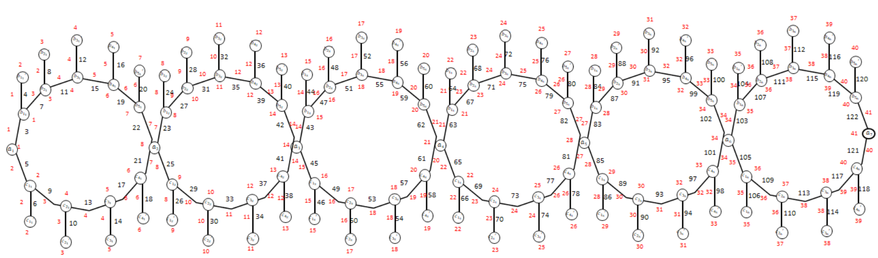

Fig. 2

Vertex and edge labels in $C_6(C_{11}^{9})$ so that tes $(C_6(C_{11}^{9})) = 41$.

3. Conclusions

We have verified that $tes(C_r(C_n^{n-2})) = \left\lceil \frac{(2n-2)r+2}{3} \right\rceil$ for $n \geq 6$. The formulas for labels of elements of the graph were presented in the theorem. In upcoming research, we are interested to investigate tvs or tes of some tadpole chain graphs.

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