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Fuzzy chromatic number of union of fuzzy graphs: An algorithm, properties and its application

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Abstract

We focus on fuzzy graphs with crisp vertex and fuzzy edge sets. A concept of the fuzzy chromatic number of these graphs based on fuzzy independent vertex set is used in this paper. A modified algorithm called a fuzzy chromatic algorithm is developed to find the fuzzy chromatic number of union of fuzzy graphs. Running time and complexity of the algorithm are also analyzed. Furthermore, we investigate some properties of the fuzzy chromatic number of union of fuzzy graphs. Finally, an application of the fuzzy chromatic number to determine the number of phases of an integrated traffic light system is proposed. We get different phases with different degrees of safety.

Keywords: Fuzzy graph; Fuzzy chromatic number; Fuzzy independent vertex set; Fuzzy chromatic algorithm; Union; Phase; Traffic light

1. Introduction

In a classical graph $G(V, E)$, connections between vertices in V are precisely known i.e., adjacent or not. Some real-life problems have been modelled using the classical graph, such as telecommunication, transportation, electricity, and traffic networks. In a traffic network, there are some factors that can result in traffic congestion at an intersection.

Therefore, an interesting research in this network is to determine the number of phases to arrange a traffic light. A phase is a part of a signal cycle with a green light allocated to a specific combination of traffic movements [1]. To model a traffic network using a classical graph, traffic flows that move from one direction to the others can be expressed as vertices. If two traffic flows may collide when they move simultaneously, then both are connected by an

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edge. The minimum number of phases required on a traffic light system is determined through the chromatic number of the classical graph.

In fact, traffic congestion at an intersection is attributable to indeterminate factors such as the number of vehicles, the length of cycle time of traffic light, situational conditions, bad attitude of drivers, etc. Hence, a tool is needed to deal with vagueness phenomena on the network. Since a fuzzy set theory had been introduced by Zadeh [2], some kinds of fuzzy graphs have been proposed by researchers. Kaufmann [3] introduced fuzzy graphs consisting of crisp vertex and fuzzy edge sets (type 1 fuzzy graphs). Moreover, Rosenfeld [3] initiated the concept of fuzzy graphs with fuzzy vertex and fuzzy edge sets (type 2 fuzzy graphs). Some researchers have generalized several basic theories of graphs. For instance, Craine [4] investigated characteristics of fuzzy interval graphs. Furthermore, Blue et al. [5] classified some types of fuzziness in graphs. Recently, Mathew et al. [6] introduced transitive blocks together with their applications in fuzzy interconnection networks. Moreover, Binu et al.[7] proposed a connectivity index of a fuzzy graph and its application to human trafficking.

Other concepts in the classical graph that have been generalized are vertex, edge, and total coloring, as well as independent sets. Vertex coloring methods in fuzzy graphs have been studied by researchers. Eslahchi and Onagh [8] constructed this method for type 2 fuzzy graphs based on strong adjacencies between vertices. In a similar way, Kishore and Sunitha [9] investigated strong coloring and chromatic number of type 2 fuzzy graphs based on strong arcs between vertices. They also gave an application of strong coloring of fuzzy graphs to solve traffic light problems. On type 1 fuzzy graphs, Bershtein and Bozhenuk [10] proposed a vertex coloring method based on maximal independent vertex sets and defined fuzzy chromatic number of fuzzy graphs through these sets. Further, Munoz et al. [11] initiated a method to color vertices of type 1 fuzzy graphs by means of α-cut coloring of the graphs with $\alpha \in [0, 1]$. Cioban [12] introduced a method for vertex coloring of fuzzy graphs through δ-fuzzy independent vertex sets with $\delta \in [0, 1]$ and the chromatic number was called δ-chromatic number. Keshavarz [13] established a method to color a type 1 fuzzy graph based on incompatibility degrees of adjacent vertices. For more research on fuzzy graphs, readers are referred to [14], [15], [16], [17], and [18].

In the year 2015, Rosyida et al. [19] presented a method to determine fuzzy chromatic number of fuzzy graphs based on δ-chromatic numbers. An algorithm to determine the number was also constructed. In the recent article, we put forward a modified fuzzy chromatic algorithm to find fuzzy chromatic number of union of fuzzy graphs that will be a useful tool in solving real-life problems. The running time and complexity of the algorithm are also verified. In continuation of the previous work in [19], we are also interested in investigating some properties of fuzzy chromatic number of union of fuzzy graphs. For example in classical graphs, there is a property that chromatic number of union of graphs is $\chi(G_1 \cup G_2) = \max\{\chi_1(G_1), \chi_2(G_2)\}$. We investigate whether or not this property could be generalized in fuzzy graphs. This is a new result in the theory of fuzzy graph coloring.

The problem of coloring fuzzy graphs is an interesting one because many real problems can be solved using the concept. Keshavarz [13] gave an application of vertex coloring on these graphs to solve a cell site assignment problem in a telecommunication network. Munoz et al. [11] discussed the application of α-cut chromatic number in arranging traffic flows at an intersection. We differ from the work by Munoz et al. in that we use fuzzy chromatic number and its application involving two intersections. We describe traffic flows on two intersections by firstly setting it into one integrated traffic light system through union of two fuzzy graphs and determine the number of phases on the system by using fuzzy chromatic number of union of fuzzy graphs. To the extent of our knowledge, these problems have not been investigated until now.

This paper is organized as follows. Section 2 discusses some basic concepts in: graph coloring, fuzzy set theory, and fuzzy graph coloring. In Section 3, an algorithm and some results on properties of fuzzy chromatic number of union of fuzzy graphs are given. In Section 4, an application of this number is proposed. Finally, conclusions are given in Section 5.

2. Preliminaries

In this section, we recall some definitions in graph theory, fuzzy set theory and fuzzy graph coloring that will be used in constructing fuzzy chromatic number of union of fuzzy graphs.

2.1. Basic concepts of vertex coloring

A graph $G(V, E)$ consists of a nonempty vertex set $V = V(G)$ and an edge set $E = E(G)$. In this paper, we consider simple, finite and undirected graphs. An edge e between two vertices u and v is presented as $e = uv$ rather than $e = (u, v)$. Two vertices connected by an edge are called adjacent. Given two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$. A graph $G(V, E) = G_1 \cup G_2$ is called union of G_1 and G_2 if $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. If $V_1 \cap V_2 = \emptyset$, then $G(V, E) = G_1 \cup G_2$ is called disjoint union of G_1 and G_2 [20].

A subset $I \subseteq V(G)$ is said to be an independent vertex set of G if $uv \notin E(G)$ for all $u, v \in I$. A mapping $f : V \to \{1, 2, \dots, k\}$ is called a k-coloring of graph $G(V, E)$ if it satisfies $f(u) \neq f(v)$ whenever $uv \in E$. As an equivalent definition, a k-coloring of graph G can be defined as a partition of V into k-independent vertex sets $I_1, I_2, \cdots, I_k$ such that the subsets I_i are nonempty ($i = 1, 2, \dots, k$), $I_i \cap I_j = \emptyset$ for $i \neq j$, and $I_1 \cup I_2 \cup \ldots \cup I_k = V$. The minimum number of colors k in the k-coloring of G is called chromatic number of G. The concept of vertex coloring of crisp graph G through partition of the vertex set of G into independent vertex sets has been generalized in fuzzy graphs by Cioban [12]. This generalization will be discussed in the next section. Moreover, the chromatic number of union of two graphs was given in [21] as follows: $\chi(G_1 \cup G_2) = \max\{\chi(G_1), \chi(G_2)\}$. In this paper, we generalize the property in fuzzy graphs by using two methods of ranking discrete fuzzy numbers.

2.2. Basic concepts of fuzzy sets

Zadeh [2] first introduced the concept of fuzzy sets in 1965. Let X be a nonempty universal set. A fuzzy set $\tilde{B}$ on X is a set $\{(x, \mu_{\tilde{B}}(x))|x \in X\}$ where $\mu_{\tilde{B}} : X \to [0, 1]$ is called a membership function of fuzzy set $\tilde{B}$. Further, we call classical sets as crisp sets. A set $S(\tilde{B}) = \{x \in X|\mu_{\tilde{B}}(x) > 0\}$ is said to be a support of fuzzy set $\tilde{B}$. A set $h(\tilde{B}) = \sup\{\mu_{\tilde{B}}(x)|x \in X\}$ is called a height of fuzzy set $\tilde{B}$. Moreover, fuzzy set $\tilde{B}$ is said to be a normal fuzzy set if $h(\tilde{B}) = 1$. Let $\alpha \in (0, 1]$, a set $B_\alpha = \{v \in X|\mu_{\tilde{B}}(v) \geq \alpha\}$ is called an α-cut of fuzzy set $\tilde{B}$.

Given two fuzzy sets $\tilde{A}_1$ and $\tilde{A}_2$ on X where their membership functions are $\mu_{\tilde{A}_1} : X \to [0, 1]$ and $\mu_{\tilde{A}_2} : X \to [0, 1]$, respectively. A symbol $\tilde{A}_1 \subseteq \tilde{A}_2$ means that fuzzy set $\tilde{A}_1$ is a subset of $\tilde{A}_2$ with $\mu_{\tilde{A}_1}(x) \leq \mu_{\tilde{A}_2}(x)$ for all $x \in X$. Furthermore, a union $\tilde{B} = \tilde{A}_1 \cup \tilde{A}_2$ is a fuzzy set on X with a membership function $\mu_{\tilde{B}} : X \to [0, 1]$ which is defined as $\mu_{\tilde{B}}(x) = \max\{\mu_{\tilde{A}_1}(x), \mu_{\tilde{A}_2}(x)\}$ for all $x \in X$. Meanwhile, an intersection $\tilde{D} = \tilde{A}_1 \cap \tilde{A}_2$ is a fuzzy set on X with a membership function $\mu_{\tilde{D}} : X \to [0, 1]$ which is defined as $\mu_{\tilde{D}}(x) = \min\{\mu_{\tilde{A}_1}(x), \mu_{\tilde{A}_2}(x)\}$ for all $x \in X$.

A fuzzy set $\tilde{A}$ on real number system $\mathbb{R}$ which satisfies the following properties: 1) $\tilde{A}$ is normal i.e., $\exists x_0 \in \mathbb{R}$ such that $\mu_{\tilde{A}}(x_0) = 1$; 2) $\tilde{A}_\alpha$ is a closed interval for every $\alpha \in (0, 1]$; and 3) the support $S(\tilde{A})$ is bounded, is called a fuzzy number [22].

The concept of discrete fuzzy numbers has been introduced in [23], [24], [25], and [26]. Let $\mathbb{R}$ be a real number system. A fuzzy set $\tilde{A}$ with a membership function $\mu : \mathbb{R} \to [0, 1]$ is said to be a discrete fuzzy number on $\mathbb{R}$ if its support is finite, i.e., there exist $a_1, a_2, \dots, a_n \in \mathbb{R}$ with $a_1 < a_2 < \ldots < a_n$ such that the support $S(\tilde{A}) = \{a_1, a_2, \dots, a_n\}$, and there are natural numbers r, s with $1 \leq r \leq s \leq n$ such that: 1) $\mu_{\tilde{A}}(a_i) = 1$ for any natural number i with $r \leq i \leq s$ (core); 2) $\mu_{\tilde{A}}(a_i) \leq \mu_{\tilde{A}}(a_j)$ for any $a_i, a_j \in \mathbb{R}$ with $1 \leq i \leq j \leq r$; 3) $\mu_{\tilde{A}}(a_i) \geq \mu_{\tilde{A}}(a_j)$ for any i, j with $s \leq i \leq j \leq n$. We have verified that fuzzy chromatic number of fuzzy graphs is a discrete fuzzy number in [19].

There are some methods for determining maximum and minimum on fuzzy numbers. Zadeh [2] gave these numbers using the extension principle. Other methods were given by some researchers such as Dubois and Prade [27], Wang and Kerre [28], and Matarazzo and Munda [29]. As claimed by Casasnovas and Riera ([24], [25]), the previous methods might not satisfy the general properties of a discrete fuzzy number. As a result, they gave a method to compare these numbers through the α-cut sets and verified that the maximum and minimum between fuzzy numbers satisfy the properties of a discrete fuzzy number. In addition, Basirzadeh [30] proposed a method to compare discrete fuzzy sets through a defuzzification process depended on a related α-cut set.

Since fuzzy chromatic number is a discrete fuzzy number, we use methods in [24] and [25] to compare the numbers. Let $\tilde{U}, \tilde{V}$ be two discrete fuzzy numbers. For each $\alpha \in [0, 1]$, the sets $U^\alpha = \{x_1^\alpha, \dots, x_m^\alpha\}$ and $V^\alpha = \{y_1^\alpha, \dots, y_n^\alpha\}$ are the α-cut sets of $\tilde{U}$ and $\tilde{V}$ respectively. Let $S(\tilde{U}) \bigvee S(\tilde{V})$ be a set $\{z|\max\{\min S(\tilde{U}), \min S(\tilde{V})\} \leq z \leq \max\{\max S(\tilde{U}), \max S(\tilde{V})\}\}$. It was defined an α-cut set

$$A^\alpha = \{z \in S(\tilde{U}) \bigvee S(\tilde{V})|\max\{\min x_1^\alpha, \min y_1^\alpha\} \leq z \leq \max\{\max x_m^\alpha, \max y_n^\alpha\}\}.$$

The maximum of $\tilde{U}$ and $\tilde{V}$ is a fuzzy set $\max\{\tilde{U}, \tilde{V}\} = \{(z, \mu_{\max\{\tilde{U},\tilde{V}\}}(z))\}$ with $\mu_{\max\{\tilde{U},\tilde{V}\}}(z) = \sup\{\alpha \in [0,1] | z \in A^\alpha\}$. Further, let $S(\tilde{U}) \bigwedge S(\tilde{V})$ be a set $\{z | \min\{\min S(\tilde{U}), \min S(\tilde{V})\} \leq z \leq \min\{\max S(\tilde{U}), \max S(\tilde{V})\}\}$. The α-cut set

$$B^\alpha = \{z \in S(\tilde{U}) \bigwedge S(\tilde{V}) | \min\{\min x_1^\alpha, \min y_1^\alpha\} \leq z \leq \min\{\max x_m^\alpha, \max y_n^\alpha\}\}.$$

The minimum of $\tilde{U}$ and $\tilde{V}$ is a fuzzy set $\min\{\tilde{U}, \tilde{V}\} = \{(z, \mu_{\min\{\tilde{U},\tilde{V}\}}(z))\}$ with

$$\mu_{\min\{\tilde{U},\tilde{V}\}}(z) = \sup\{\alpha \in [0,1] | z \in B^\alpha\}.$$

On the other hand, we also use a method to compare discrete fuzzy sets by using a defuzzification which depends on a decision level higher than α as given in [30]. Given a universal set X and $v_1, v_2, \ldots, v_n \in X$. Let $\tilde{A} = \{(v_i, \mu_{\tilde{A}}(v_i))\}$ for $i \in \{1, 2, \ldots, n\}$ be a discrete fuzzy set. Given a level $\alpha \in [0, 1]$. A quantity $Q_\alpha(\tilde{A})$ represents a defuzzification of fuzzy set $\tilde{A}$ which depends on a decision level higher than α. It is assumed that $v_1 \leq v_2 \leq \ldots \leq v_n$. A method to calculate crisp value $Q_\alpha(\tilde{A})$ is as follows:

1. If all values of membership functions in $\tilde{A}$ are increasing i.e., $\mu_{\tilde{A}}(v_1) \leq \mu_{\tilde{A}}(v_2) \leq \ldots \leq \mu_{\tilde{A}}(v_n)$ and $0 \leq \alpha \leq \mu_{\tilde{A}}(v_n)$, then $Q_\alpha(\tilde{A})$ is defined as:

$$Q_\alpha(\tilde{A}) = v_1(\mu_{\tilde{A}}(v_1) - \alpha) + \sum_{i=2}^{n} v_i(\mu_{\tilde{A}}(v_i) - \mu_{\tilde{A}}(v_{i-1})). \tag{1}$$

2. If all values of membership functions in $\tilde{A}$ are decreasing i.e., $\mu_{\tilde{A}}(v_1) \geq \mu_{\tilde{A}}(v_2) \geq \ldots \geq \mu_{\tilde{A}}(v_n)$ and $0 \leq \alpha \leq \mu_{\tilde{A}}(v_n)$, then

$$Q_\alpha(\tilde{A}) = v_n(\mu_{\tilde{A}}(v_n) - \alpha) + \sum_{i=1}^{n-1} x_i(\mu_{\tilde{A}}(v_i) - \mu_{\tilde{A}}(v_{i+1})).$$

3. For $0 \leq \alpha \leq \min\{\mu_{\tilde{A}}(v_1), \mu_{\tilde{A}}(v_2), \ldots, \mu_{\tilde{A}}(v_n)\}$: if there is $v_t \in \tilde{A}$ such that $\mu_{\tilde{A}}(v_1) \leq \mu_{\tilde{A}}(v_2) \leq \ldots \leq \mu_{\tilde{A}}(v_t)$ and $\mu_{\tilde{A}}(v_t) \geq \mu_{\tilde{A}}(v_{t+1}) \geq \ldots \geq \mu_{\tilde{A}}(v_n)$, then

$$\begin{aligned} Q_\alpha(\tilde{A}) = v_1(\mu_{\tilde{A}}(v_1) - \alpha) + \sum_{i=2}^{t} v_i(\mu_{\tilde{A}}(v_i) - \mu_{\tilde{A}}(v_{i-1})) + \\ v_n(\mu_{\tilde{A}}(v_n) - \alpha) + \sum_{i=t}^{n-1} v_i(\mu_{\tilde{A}}(v_i) - \mu_{\tilde{A}}(v_{i+1})). \end{aligned}$$

Let $\tilde{A}$ and $\tilde{B}$ be two arbitrary discrete fuzzy sets. Given $\alpha \in [0, 1]$. Basirzadeh et al. [30] used defuzzifications $Q_\alpha(\tilde{A})$ and $Q_\alpha(\tilde{B})$ to compare discrete fuzzy sets $\tilde{A}$ and $\tilde{B}$ as follows:

$$\begin{aligned} &\tilde{A} \leq_\alpha \tilde{B} \Leftrightarrow Q_\alpha(\tilde{A}) \leq Q_\alpha(\tilde{B}); \ \tilde{A} =_\alpha \tilde{B} \Leftrightarrow Q_\alpha(\tilde{A}) = Q_\alpha(\tilde{B}); \\ &\tilde{A} \geq_\alpha \tilde{B} \text{ if and only if } Q_\alpha(\tilde{A}) \geq Q_\alpha(\tilde{B}); \end{aligned} \tag{2}$$

where $\tilde{A} \leq_\alpha \tilde{B}$ means $\tilde{A}$ is less than or equal to $\tilde{B}$ at a decision level higher than α. If we use different decision level α, then we may get different ranking.

2.3. Basic concepts of fuzzy graph coloring

In this section, we discuss some terminologies in fuzzy graphs and fuzzy graph coloring as cited from [10], [12], [15], [17], and [18].

Let V be a finite nonempty set and $E \subseteq V \times V$. A fuzzy graph which has a crisp vertex set V and a fuzzy edge set $\tilde{E}$ with membership function $\mu : V \times V \to [0, 1]$ is denoted as $\tilde{G}(V, \tilde{E})$. Meanwhile, a fuzzy graph which consists of a fuzzy vertex set $\tilde{V}$ with membership function $\sigma : V \to [0, 1]$ and a fuzzy edge set $\tilde{E}$ with membership function

$\mu : V \times V \to [0, 1]$ such that $\mu(uv) \le \min\{\sigma(u), \sigma(v)\}$ for all $u, v \in V$ is denoted as $\tilde{G}(\tilde{V}, \tilde{E})$. In this paper, we deal with fuzzy graph $\tilde{G}(V, \tilde{E})$. Further, we call classical graph $G(V, E)$ as a crisp graph. The underlying graph of fuzzy graph $\tilde{G}(V, \tilde{E})$ is a crisp graph $G^*(V^*, E^*)$ with $V^* = V$ and $E^* = \{uv|\mu(uv) > 0, u, v \in V\}$. Let $\tilde{G}_1(V_1, \tilde{E}_1)$ be a fuzzy graph with crisp vertex set V_1 and fuzzy edge set $\tilde{E}_1$ with a membership function $\mu_{\tilde{E}_1} : V_1 \times V_1 \to [0, 1]$. Fuzzy graph $\tilde{G}_1$ is said to be a fuzzy subgraph of $\tilde{G}(V, \tilde{E})$ if $V_1 \subseteq V$ and $\tilde{E}_1 \subseteq \tilde{E}$.

In this paper, we rewrite the notion of union of fuzzy graphs cited from [15] as presented in Definition 1.

Definition 1. Let V_1 and V_2 be finite nonempty sets. Let $\tilde{G}_1(V_1, \tilde{E}_1)$ and $\tilde{G}_2(V_2, \tilde{E}_2)$ be two fuzzy graphs where the membership functions of $\tilde{E}_1$ and $\tilde{E}_2$ are μ_1 and μ_2, respectively. A union of $\tilde{G}_1$ and $\tilde{G}_2$ is a fuzzy graph $\tilde{G}(V, \tilde{E}) = \tilde{G}_1 \cup \tilde{G}_2$ which has a vertex set $V = V_1 \cup V_2$ and a fuzzy edge set $\tilde{E} = \tilde{E}_1 \cup \tilde{E}_2$ with

$$\mu_{\tilde{E}}(uv) = \max\{\mu_1(uv), \mu_2(uv)\},$$

for all $u, v \in V$. If $V_1 \cap V_2 = \emptyset$, then $\tilde{G}(V, \tilde{E}) = \tilde{G}_1 \cup \tilde{G}_2$ is called disjoint union of fuzzy graphs $\tilde{G}_1$ and $\tilde{G}_2$.

The concept of fuzzy independent vertex set based on $\delta \in [0, 1]$ proposed by Cioban [12] is presented in Definition 2.

Definition 2. Let $\tilde{G}(V, \tilde{E})$ be a fuzzy graph. Given $\delta \in [0, 1]$. A fuzzy independent vertex set $S \subseteq V$ is defined as a set where $\mu(uv) \le \delta$ for all $u, v \in S$. The fuzzy independent vertex set S is also called δ-fuzzy independent vertex set denoted by S^δ.

A degree of independence of fuzzy independent vertex set S^δ has been given in [18] as follows: $\alpha(S^\delta) = 1 - \max\{\mu(xy)|x, y \in S^\delta\}$. When $\delta = 0$, fuzzy independent vertex set S^δ has degree of independence $\alpha(S^\delta) = 1$ and it becomes a crisp independent vertex set of underlying crisp graph G^*. When $\delta = 1$, fuzzy independent vertex set S^δ has degree $\alpha(S^\delta) = 0$. In other words, fuzzy independent vertex set S^δ on $\delta = 1$ does not coincide with a crisp independent vertex set of underlying crisp graph G^*. In Definition 3, the concept of k-coloring of fuzzy graph $\tilde{G}$ has been rewritten based on δ-fuzzy independent vertex sets given by Cioban [12].

Definition 3. Given $\delta \in [0, 1]$. A k-coloring of fuzzy graph $\tilde{G}(V, \tilde{E})$ is defined as a function $f : V \to \{1, 2, \dots, k\}$ such that $f(u) = f(v)$ if $\mu(uv) \le \delta$. The minimum number k in the k-coloring of $\tilde{G}$ is called δ-chromatic number of $\tilde{G}$ and denoted by $\chi^\delta(\tilde{G})$.

Based on Definition 3, a k-coloring of fuzzy graph $\tilde{G}(V, \tilde{E})$ can be obtained by partitioning vertex set V into k-sets which are δ-fuzzy independent vertex sets $\{S_1^\delta, \dots, S_k^\delta\}$ such that $S_i^\delta \cap S_j^\delta = \emptyset$ for all $i \ne j$ and $S_1^\delta \cup \dots \cup S_k^\delta = V$. Furthermore, an approach to determine fuzzy chromatic number of fuzzy graph $\tilde{G}(V, \tilde{E})$ which is cited from [19], is presented in Definition 4.

Definition 4. Let $\tilde{G}(V, \tilde{E})$ be a fuzzy graph with n vertices. A fuzzy chromatic number of $\tilde{G}$, denoted by $\tilde{\chi}(\tilde{G})$, is a fuzzy set

$$\tilde{\chi}(\tilde{G}) = \{(k, L_{\tilde{\chi}}(k))|k = 1, \dots, n\}$$

where the value $L_{\tilde{\chi}}(k) = \max\{1 - \delta|\delta \in [0, 1], \chi^\delta(\tilde{G}) = k\}$ represents a degree of membership of number k in fuzzy chromatic number $\tilde{\chi}$.

Note that if $\delta \in [0, 1]$ does not exist such that $\chi^\delta(\tilde{G}) = k$, then we define $L_{\tilde{\chi}}(k) = L_{\tilde{\chi}}(k - 1)$ and this is consistent to a property of fuzzy chromatic number in [10]. Let us consider the following example.

Example 1. Given fuzzy graph $\tilde{G}(V, \tilde{E})$ in Fig. 1 which has vertex set $V = \{AB, DC, AD, CB, CD, DB\}$ and fuzzy edge set $\tilde{E}$ consists of 7 edges. It can be observed that vertices AD and CD are δ-fuzzy independent for $\delta = 0.48$. Meanwhile, vertices AD and CB are δ-fuzzy independent on $\delta = 0$ and are associated with crisp independent vertices of underlying crisp graph G^*.

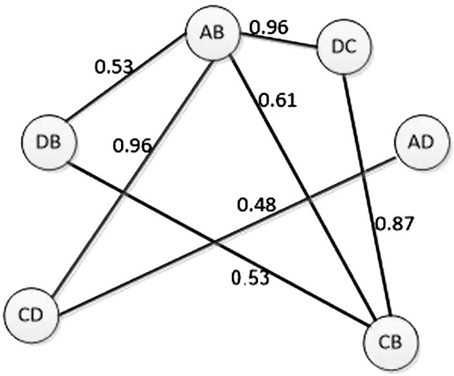

Fig. 1. A fuzzy graph with 6 vertices.

Table 1
Partitions on V and δ-chromatic numbers of $\tilde{G}$.

δ	Partitions on V	$\chi^{\delta}(\tilde{G})$
0	$\{\{DC, AD, DB\}, \{CD, CB\}, \{AB\}\}$	3
0.48	$\{\{DC, AD, DB\}, \{CD, CB\}, \{AB\}\}$	3
0.53	$\{\{DC, AD, DB\}, \{CD, CB\}, \{AB\}\}$	3
0.61	$\{\{AB, DB, CB\}, \{DC, AD, CD\}\}$	2
0.87	$\{\{AB, DB, CB\}, \{DC, AD, CD\}\}$	2
0.96	V	1

Some partitions of V and δ-chromatic numbers of $\tilde{G}$ are presented in Table 1. Based on Definition 4 and Table 1, we obtain fuzzy chromatic number of $\tilde{G}$ as follows:

$$\tilde{\chi}(\tilde{G}) = \{(k, L_{\tilde{\chi}}(k)) | k = 1, \dots, n\} = \{(1, 0.04), (2, 0.39), (3, 1), (4, 1), (5, 1), (6, 1)\}.$$

In general, δ-chromatic numbers decrease when values of δ increase [19]. Also, degrees $L_{\tilde{\chi}}(k)$ in fuzzy chromatic number $\tilde{\chi}$ decrease when values of δ increase.

3. Main results

This section consists of two subsections. We propose a modified fuzzy chromatic algorithm to determine fuzzy chromatic number of union of fuzzy graphs in the first subsection. Later, the running time and complexity of the algorithm are shown. In the second subsection, we generalize the chromatic number of union of crisp graphs into fuzzy chromatic number of union of fuzzy graphs by means of Definition 4. The result is presented in Theorem 1. Some related properties are also investigated as presented in Theorems 2 and 3. Moreover, some properties of fuzzy chromatic number of a fuzzy subgraph are provided in Lemma 1 and Theorem 4. By using Theorem 1, we are able to determine fuzzy chromatic number of union of fuzzy graphs in a simple way.

3.1. Modified fuzzy chromatic algorithm for union of fuzzy graphs

In order to determine fuzzy chromatic number of union of fuzzy graphs, an algorithm (with its complexity), the flowchart (in Fig. 2), and the running time (in Fig. 3) are proposed. The algorithm is structured based on the concept of fuzzy chromatic number in Definition 4. The first step is to divide vertex set V into fuzzy independent vertex sets which depend on $\delta \in [0, 1]$ as mentioned in Definition 2. The second step is to find δ-chromatic numbers of fuzzy graph $\tilde{G}$ based on the partitions used in the first step. Lastly, the third step is to determine fuzzy chromatic number of union of fuzzy graphs based on the δ-chromatic numbers.

Algorithm for fuzzy chromatic number of union of fuzzy graphs.	
Input n_1, n_2 number of vertices	$O(1)$
$V_1 = 1:n_1$; $V_2 = 1:n_2$ % $V_1 = \{v_i \| i = 1, \dots, n_1\}$, $V_2 = \{v'_i \| i = 1, \dots, n_2\}$	$O(1)$
$V = V_1 \cup V_2$	$O(1)$
E_1=combination(n_1,2) % $E_1 = \{e_i \| i = 1, \dots, m_1, e_i = v_j v_k, j \neq k, j, k = 1, \dots, n_1\}$	$O(1)$
E_2=combination(n_2,2) % $E_2 = \{e_i \| i = 1, \dots, m_2, \}$	$O(1)$
for i =1 to m_1	$O(m)$
μ_1(i)= input weight of edge-i in E_1 % $\mu_1 = \{\mu_i \| i, \dots, m_1, \mu_i = \mu_1(e_i)\}$	$O(m)$
end	$O(m)$
for i =1 to m_2	$O(m)$
μ_2(i)= input weight of edge-i in E_2 % $\mu_2 = \{\mu_i \| i, \dots, m_2, \mu_i = \mu_2(e_i)\}$	$O(m)$
end	$O(m)$
$E = E_1 \cup E_2$; $\mu(e_i) = \max\{\mu_1(e_i), \mu_2(e_i)\}$ % $E = \{e_i \| i = 1, \dots, m\}$	$O(m)$
$\delta =$ **sort**(μ)	$O(1)$
for ind = 1 to m	$O(m)$
d=δ(ind); s=0	$O(m)$
for h = 1 to m	$O(m^2)$
if μ(h) $\leq$ d	$O(m^2)$
s=s+1	$O(m^2)$
S(s,:)=E(h,:), % $S = \{e_i \| i = 1, \dots, p, \mu_i \leq d, p \leq m\}$	$O(m^2)$
end	$O(m^2)$
end	$O(m^2)$
for i = 1 to p	$O(m^2)$
T=V-S(i,:); %$T = \{v_i \| i = 1, \dots, q, v_i \notin S(i,:), q \leq m\}$	$O(m^2)$
P=S(i,:)	$O(m^2)$
for j =1 to q	$O(m^3)$
[r,c] = size(P)	$O(m^3)$
t=1	$O(m^3)$
while t $\leq$ r	$O(m^4)$
U = P(t,:); u = 1	$O(m^4)$
score = 0	$O(m^4)$
while u $\leq$ length(U)	$O(m^5)$
Z=sort([U(u) T(j)])	$O(m^5)$
if μ(Z) $\leq$ d	$O(m^5)$
score=score+1	$O(m^5)$
end	$O(m^5)$
u=u+1	$O(m^5)$
end	$O(m^5)$
if score = length(U)	$O(m^4)$
P(t,c+1) = T(j)	$O(m^4)$
r=t	$O(m^4)$
else	$O(m^4)$
if t = r	$O(m^4)$
P(t+1,1)=T(j)	$O(m^4)$
r = t	$O(m^4)$
end	$O(m^4)$
end	$O(m^4)$
if P = V	$O(m^4)$
r = t	$O(m^4)$
end	$O(m^4)$
t=t+1	$O(m^4)$
end	$O(m^3)$
end	$O(m^3)$
Q(:,:,i) = P	$O(m^2)$
end	$O(m^2)$
PartitionSet = Q ; crom = number of Partition in Partition Set	$O(m)$
D(ind) = d; Xc(ind) = min(crom)	$O(m)$
end	$O(m)$
$X = \{k \| k \in$ Xc$\}$; L = max{1 - D|$X = k$}; **Print** $\chi = \{k, L(k)\}$	$O(1)$

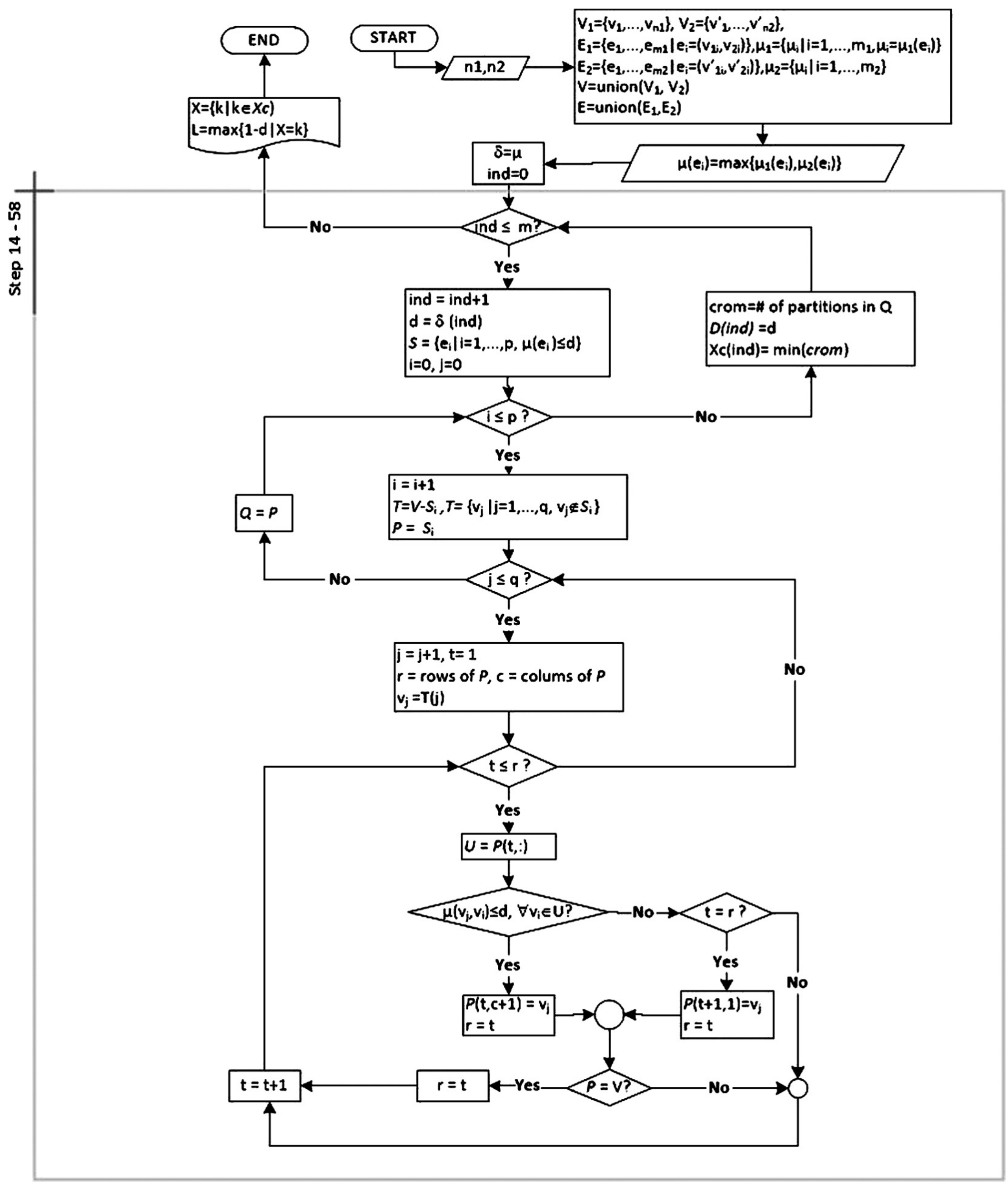

Fig. 2. A flowchart to find fuzzy chromatic number of union of fuzzy graphs.

Fig. 3 illustrates the plot of running time of the algorithm applied on Matlab R2016a. The running time is elapsed time for main looping (step 14-58) on the algorithm or flowchart in Fig. 2. It shows a "worst case" when membership degrees are assigned randomly at interval (0,1). The number of vertices varies from 3 to 7 where the membership degrees of edges are generated randomly and the number of trials is set to 150. It can be observed that the algorithm has a constant running time on all trials. Moreover the higher the number of vertices, the longer the running time.

Let us consider the fuzzy graph in Fig. 1 given in Example 1. By employing fuzzy chromatic algorithm, we get output of fuzzy chromatic number $\tilde{\chi}(\tilde{G}) = \{(1, 0.04), (2, 0.39), (3, 1), (4, 1), (5, 1), (6, 1)\}$ as presented in Fig. 4.

In the next section, we propose fuzzy chromatic number of union of fuzzy graphs which will be used later in an application.

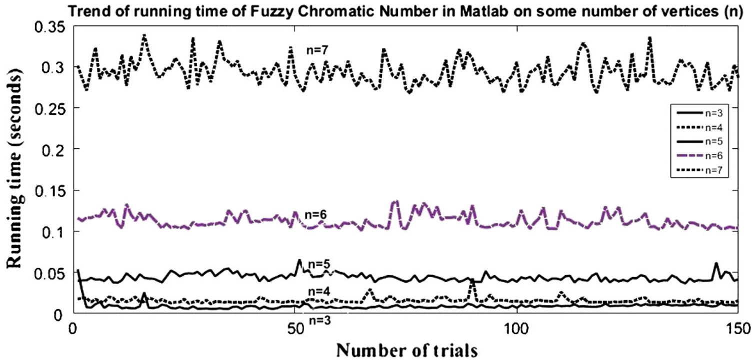

Fig. 3. Running time of modified fuzzy chromatic algorithm.

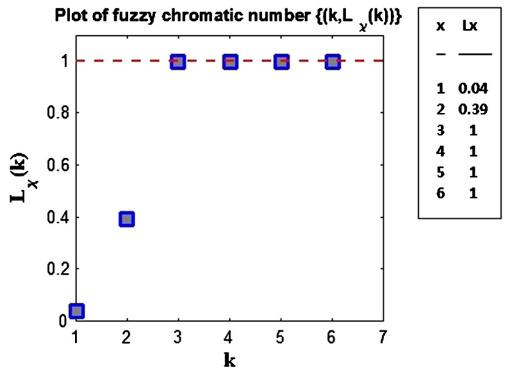

Fig. 4. An output of fuzzy chromatic algorithm for fuzzy graph in Fig. 1.

3.2. Fuzzy chromatic number of union of fuzzy graphs

In this paper, union of fuzzy graphs is used in modelling two consecutive traffic lights into one integrated traffic light system. After that, fuzzy chromatic number of union of fuzzy graphs is needed to determine the number of phases used in setting up traffic lights on the integrated system. One of new results in this subsection is a generalization of crisp chromatic number of union of two crisp graphs into fuzzy chromatic number of union of two fuzzy graphs as presented in Theorem 1. Meanwhile, the result given in [19] which is construction of a fuzzy chromatic number concept of fuzzy graphs is used in getting the result in Theorem 1. Other new results are comparisons between fuzzy chromatic numbers $\tilde{\chi}_1(\tilde{G}_1)$, $\tilde{\chi}_2(\tilde{G}_2)$, and fuzzy chromatic number of union $\tilde{\chi}(\tilde{G}_1 \cup \tilde{G}_2)$ in Theorem 2 and Theorem 3 by using two methods of comparisons between discrete fuzzy numbers.

Theorem 1. *Let $\tilde{G}_1(V_1, \tilde{E}_1)$ and $\tilde{G}_2(V_2, \tilde{E}_2)$ be two fuzzy graphs with $|V_1| = n_1$ and $|V_2| = n_2$. Let $\tilde{\chi}_1$ and $\tilde{\chi}_2$ be fuzzy chromatic numbers of $\tilde{G}_1$ and $\tilde{G}_2$, respectively. If $\tilde{G} = \tilde{G}_1 \cup \tilde{G}_2$, then $\tilde{\chi}(\tilde{G}) = \{(k, L_{\tilde{\chi}}(k))\}$ where*

$$L_{\tilde{\chi}}(k) = \begin{cases} \min\{L_{\tilde{\chi}_1}(k), L_{\tilde{\chi}_2}(k)\}, & \text{if} \quad 1 \le k \le \min\{n_1, n_2\}, \\ L_{\tilde{\chi}_1}(k), & \text{if} \quad \min\{n_1, n_2\} < k \le n_1 = \max\{n_1, n_2\}, \\ L_{\tilde{\chi}_2}(k), & \text{if} \quad \min\{n_1, n_2\} < k \le n_2 = \max\{n_1, n_2\}, \\ 1, & \text{if} \quad \max\{n_1, n_2\} < k \le n = n_1 + n_2. \end{cases}$$

Proof. Let $V_1 = \{v_1, v_2, \ldots, v_{n_1}\}$ and $V_2 = \{u_1, u_2, \ldots, u_{n_2}\}$.

Let $\tilde{\chi}_1 = \{(k, L_{\tilde{\chi}_1}(k)) | k = 1, 2, \ldots, n_1\}$ and $\tilde{\chi}_2 = \{(k, L_{\tilde{\chi}_2}(k)) | k = 1, 2, \ldots, n_2\}$ be fuzzy chromatic numbers of $\tilde{G}_1$ and $\tilde{G}_2$, respectively. We consider the following 3 cases:

Case 1. For $1 \le k \le \min\{n_1, n_2\}$.

It follows from Definition 4 that for each k, there exists a $\delta_k \in [0, 1]$ such that we can form a partition $\{S_1^{\delta_k}, S_2^{\delta_k}, \dots, S_k^{\delta_k}\}$ on V_1 which gives $\chi^{\delta_k}(\tilde{G}_1) = k$ and $L_{\tilde{\chi}_1}(k) = 1 - \delta_k$. In similar way, there exists a $\delta_{k'} \in [0, 1]$ such that we can construct a partition $\{P_1^{\delta_{k'}}, P_2^{\delta_{k'}}, \dots, P_k^{\delta_{k'}}\}$ on V_2 which gives $\chi^{\delta_{k'}}(\tilde{G}_2) = k$ and $L_{\tilde{\chi}_2}(k) = 1 - \delta_{k'}$.

Further, we choose $\delta = \max\{\delta_k, \delta_{k'}\}$ and construct a partition

$$Q = \{Q_1^{\delta} = S_1^{\delta_k} \cup P_1^{\delta_{k'}}, Q_2^{\delta} = S_2^{\delta_k} \cup P_2^{\delta_{k'}}, \dots, Q_k^{\delta} = S_k^{\delta_k} \cup P_k^{\delta_{k'}}\}$$

which gives $\chi^{\delta}(\tilde{G}_1 \cup \tilde{G}_2) = k$ and

$$L_{\tilde{\chi}}(k) = 1 - \delta = 1 - \max\{\delta_k, \delta_{k'}\}. \tag{3}$$

Thus, $L_{\tilde{\chi}}(k) = \min\{L_{\tilde{\chi}_1}(k), L_{\tilde{\chi}_2}(k)\}$ can be obtained directly from (3).

Case 2. For $\min\{n_1, n_2\} < k \le \max\{n_1, n_2\}$.

Without loss of generality, assume that $\max\{n_1, n_2\} = n_1$. We consider two subcases as follows:

1. Maximum clique of underlying graph $(G_1 \cup G_2)^*$ is contained in G_1^*. In this case, there exists $\delta_1 \in [0, 1]$ such that we can construct a partition $\{S_1^{\delta_1}, S_2^{\delta_1}, \dots, S_k^{\delta_1}\}$ on V_1 which gives $\chi^{\delta_1}(\tilde{G}_1) = k$ and $L_{\tilde{\chi}_1}(k) = 1 - \delta_1$. Further, we choose $\delta = \delta_1$ and construct a partition $Q = \{Q_1^{\delta_1} = S_1^{\delta_1} \cup \{u_1\}, Q_2^{\delta_1} = S_2^{\delta_1} \cup \{u_2\}, \dots, Q_{n_2}^{\delta_1} = S_{n_2}^{\delta_1} \cup \{u_{n_2}\}, Q_{n_2+1}^{\delta_1} = S_{n_2+1}^{\delta_1}, \dots, Q_k^{\delta_1} = S_k^{\delta_1}\}$ on $V = V_1 \cup V_2$ which gives $\chi^{\delta}(\tilde{G}) = k$ and $L_{\tilde{\chi}}(k) = L_{\tilde{\chi}_1}(k) = 1 - \delta_1$.
2. Maximum clique of underlying graph $(G_1 \cup G_2)^*$ is contained in G_2^* and $\max\{n_1, n_2\} = n_1$. In this case, we obtain $L_{\tilde{\chi}}(k) = 1 = L_{\tilde{\chi}_1}(k) = L_{\tilde{\chi}_2}(k)$ for $\min\{n_1, n_2\} < k \le \max\{n_1, n_2\}$.

Case 3. For $\max\{n_1, n_2\} < k \le n = n_1 + n_2$.

Since $n_1 = \max\{n_1, n_2\}$, we get $L_{\tilde{\chi}}(n_1) = 1$. It follows from Theorem 4.3 in [19] that $L_{\tilde{\chi}}(i) = L_{\tilde{\chi}}(j) = 1$ for any i, j with $n_1 \le i \le j \le n$. If we assume $\max\{n_1, n_2\} = n_2$, we will get the result in a similar way. □

As a consequence of Theorem 1, we give a remark as follows.

Remark 1. Let $\tilde{G}_1(V_1, \tilde{E}_1)$ and $\tilde{G}_2(V_2, \tilde{E}_2)$ be two fuzzy graphs.

If $\tilde{G}_1 \subseteq \tilde{G}_2$, then $\tilde{G} = \tilde{G}_1 \cup \tilde{G}_2 = \tilde{G}_2$. Hence, fuzzy chromatic number of union $\tilde{G} = \tilde{G}_1 \cup \tilde{G}_2$ is $\tilde{\chi}(\tilde{G}) = \{(k, L_{\tilde{\chi}}(k))\}$, where $L_{\tilde{\chi}}(k) = L_{\tilde{\chi}_2}(k)$, for $1 \le k \le n = |V_2|$.

In crisp graphs, if G_1 and G_2 have chromatic numbers χ_1 and χ_2, respectively, then the chromatic number of union $G = G_1 \cup G_2$ is $\chi = \max\{\chi_1, \chi_2\}$. We are interested in investigating this problem on fuzzy graphs, as stated in Theorem 2. In this paper, we use two methods to compare fuzzy chromatic numbers.

Firstly, we make use of the method to compare discrete fuzzy sets given in [30]. The symbol $Q_\alpha(\tilde{\chi})$ stands for defuzzification of fuzzy set $\tilde{\chi}$ for a decision level higher than α.

Theorem 2. *Let $\tilde{G}_1(V_1, \tilde{E}_1)$ and $\tilde{G}_2(V_2, \tilde{E}_2)$ be two fuzzy graphs with fuzzy chromatic numbers $\tilde{\chi}_1$ and $\tilde{\chi}_2$, respectively. Let $\tilde{G}(V, \tilde{E}) = \tilde{G}_1 \cup \tilde{G}_2$ be a union of $\tilde{G}_1$ and $\tilde{G}_2$ with $|V_1| = n_1$, $|V_2| = n_2$, and $|V| = n_1 + n_2$.*

If $\tilde{G}(V, \tilde{E}) = \tilde{G}_1 \cup \tilde{G}_2$ has fuzzy chromatic number $\tilde{\chi}(\tilde{G})$ as given in Theorem 1, then $Q_\alpha(\tilde{\chi}) \ge \max\{Q_\alpha(\tilde{\chi}_1), Q_\alpha(\tilde{\chi}_2)\}$ for all decision levels $\alpha \in [0, 1]$.

Proof. Let $0 \le \alpha \le 1$ and $n = n_1 + n_2$. Without loss of generality, we assume $\max\{n_1, n_2\} = n_1$. Since the values of membership functions of the fuzzy chromatic numbers $\tilde{\chi}$, $\tilde{\chi}_1$, and $\tilde{\chi}_2$ are increasing, it follows from equation (1) that:

$$\begin{aligned} Q_\alpha(\tilde{\chi}_1) = & \ 1\left(L_{\tilde{\chi}_1}(1) - \alpha\right) + 2\left(L_{\tilde{\chi}_1}(2) - L_{\tilde{\chi}_1}(1)\right) + 3\left(L_{\tilde{\chi}_1}(3) - L_{\tilde{\chi}_1}(2)\right) + \\ & \ \dots + n_1\left(L_{\tilde{\chi}_1}(n_1) - L_{\tilde{\chi}_1}(n_1 - 1)\right), \\ = & \ -L_{\tilde{\chi}_1}(1) - \alpha - L_{\tilde{\chi}_1}(2) - L_{\tilde{\chi}_1}(3) - \dots - L_{\tilde{\chi}_1}(n_1 - 1) + n_1(1), \end{aligned} \tag{4}$$

$$\begin{aligned}Q_\alpha(\tilde{\chi}_2) = &\ 1\left(L_{\tilde{\chi}_2}(1)-\alpha\right)+2\left(L_{\tilde{\chi}_2}(2)-L_{\tilde{\chi}_2}(1)\right)+3\left(L_{\tilde{\chi}_2}(3)-L_{\tilde{\chi}_2}(2)\right)+\\ &\ \ldots+n_2\left(L_{\tilde{\chi}_2}(n_2)-L_{\tilde{\chi}_2}(n_2-1)\right),\\ = &\ -L_{\tilde{\chi}_2}(1)-\alpha-L_{\tilde{\chi}_2}(2)-L_{\tilde{\chi}_2}(3)-\ldots-L_{\tilde{\chi}_2}(n_2-1)+n_2(1),\\ Q_\alpha\ (\tilde{\chi}) = &\ 1\left(L_{\tilde{\chi}}(1)-\alpha\right)+2\left(L_{\tilde{\chi}}(2)-L_{\tilde{\chi}}(1)\right)+3\left(L_{\tilde{\chi}}(3)-L_{\tilde{\chi}_2}(2)\right)+\ldots\\ &+n_1\left(L_{\tilde{\chi}}(n_1)-L_{\tilde{\chi}}(n_1-1)\right)+(n_1+1)\left(L_{\tilde{\chi}}(n_1+1)-L_{\tilde{\chi}}(n_1)\right)+\\ &\ \ldots+n(L_{\tilde{\chi}}(n)-L_{\tilde{\chi}}(n-1)).\end{aligned} \tag{5}$$

Since $\max\{n_1,n_2\}=n_1$ and $L_{\tilde{\chi}}(k)=1$ for $n_1<k\le n$, we get

$$Q_\alpha(\tilde{\chi})=-L_{\tilde{\chi}}(1)-\alpha-L_{\tilde{\chi}}(2)-L_{\tilde{\chi}}(3)-\ldots-L_{\tilde{\chi}}(n_1-1)+n_1(1)+\ldots+n(0).$$

Based on Theorem 1, it is obvious that

$$\begin{aligned}Q_\alpha(\tilde{\chi})=&-\alpha+\max\{-L_{\tilde{\chi}_1}(1),-L_{\tilde{\chi}_2}(1)\}+\max\{-L_{\tilde{\chi}_1}(2),-L_{\tilde{\chi}_2}(2)\}+\\ &\max\{-L_{\tilde{\chi}_1}(3),-L_{\tilde{\chi}_2}(3)\}+\ldots+\max\{-L_{\tilde{\chi}_1}(n_1-1),-L_{\tilde{\chi}_2}(n_1-1)\}+n_1.\end{aligned}$$

Thus, $Q_\alpha(\tilde{\chi})\ge\max\{Q_\alpha(\tilde{\chi}_1),Q_\alpha(\tilde{\chi}_2)\}$. If we assume $\max\{n_1,n_2\}=n_2$, we will get the result in a similar way and the theorem is proved. □

Secondly, we employ the method to compare discrete fuzzy numbers given in [24] and [25], as stated in Theorem 3.

Theorem 3. *Let $\tilde{G}_1(V_1,\tilde{E}_1)$ and $\tilde{G}_2(V_2,\tilde{E}_2)$ be two fuzzy graphs with $V_1\cap V_2=\emptyset$. The fuzzy chromatic numbers of $\tilde{G}_1$ and $\tilde{G}_2$ are $\tilde{\chi}_1$ and $\tilde{\chi}_2$, respectively. If $\tilde{G}=\tilde{G}_1\cup\tilde{G}_2$ is the union of $\tilde{G}_1$ and $\tilde{G}_2$, then fuzzy chromatic number $\tilde{\chi}(\tilde{G})$ satisfies*

$$\tilde{\chi}(\tilde{G})\supseteq\max\{\tilde{\chi}_1,\tilde{\chi}_2\}.$$

Proof. Given $\tilde{\chi}_1=\{(k,L_{\tilde{\chi}_1}(k))\}$ and $\tilde{\chi}_2=\{(k,L_{\tilde{\chi}_2}(k))\}$.

Let $\alpha\in[0,1]$, $\chi_1^\alpha=\{x_1^\alpha,\ldots,x_m^\alpha\}$ and $\chi_2^\alpha=\{y_1^\alpha,\ldots,y_n^\alpha\}$ are α-cut sets of $\tilde{\chi}_1$ and $\tilde{\chi}_2$ respectively. It is clear that $S(\tilde{\chi}_1)=\{1,2,\ldots,n_1\}$ and $S(\tilde{\chi}_2)=\{1,2,\ldots,n_2\}$. We get $S(\tilde{\chi}_1)\bigvee S(\tilde{\chi}_2)=\{1,2,\ldots,\max\{n_1,n_2\}\}$.

For each $k\in\{1,2,\ldots,\min\{n_1,n_2\}\}$, $\alpha_{\min}(k)=\min\{L_{\tilde{\chi}_1}(k),L_{\tilde{\chi}_2}(k)\}$, and the set $\alpha_{\max}(k)=\max\{L_{\tilde{\chi}_1}(k),L_{\tilde{\chi}_2}(k)\}$. Without loss of generality, assume that $\min\{L_{\tilde{\chi}_1}(k),L_{\tilde{\chi}_2}(k)\}=L_{\tilde{\chi}_1}(k)$ or vice versa. The $\alpha_{\min}$ and $\alpha_{\max}$-cut sets are

$$\chi_1^{\alpha_{\min}}=\{k,k+1,\ldots,n_1\}=\chi_2^{\alpha_{\min}};\ \chi_2^{\alpha_{\max}}=\{k',k'+1,\ldots,n_2\}\text{ and }\chi_1^{\alpha_{\max}}=\{k+1,k+2,\ldots,n_2\}\text{ with }k'\le k.$$

Therefore,

$$\begin{aligned}A^{\alpha_{\min}}&=\{z\in\{1,2,\ldots,\max\{n_1,n_2\}\}|k\le z\le\max\{n_1,n_2\}\}\\ &=\{k,k+1,\ldots\max\{n_1,n_2\}\}.\\ A^{\alpha_{\max}}&=\{z\in\{1,2,\ldots,\max\{n_1,n_2\}\}|k+1\le z\le\max\{n_1,n_2\}\}\\ &=\{k+1,\ldots\max\{n_1,n_2\}\}.\end{aligned}$$

Let $\max\{\tilde{\chi}_1,\tilde{\chi}_2\}=\{(z,\mu_{\max\{\tilde{\chi}_1,\tilde{\chi}_2\}}(z))\}$. We obtain

$$\mu_{\max\{\tilde{\chi}_1,\tilde{\chi}_2\}}(k)=\alpha_{\min}=\min\{L_{\tilde{\chi}_1}(k),L_{\tilde{\chi}_2}(k)\}.$$

Without loss of generality, we assume that $\min\{n_1,n_2\}=n_1$ and the value $\alpha'=L_{\tilde{\chi}_2}(k)$ for $n_1+1\le k\le n_2$. It is obvious that $\chi_1^{\alpha'}=\{n_1\}$ and $\chi_2^{\alpha'}=\{k,\ldots,n_2\}$.

We get

$$A^{\alpha'}=\{z\in\{1,2,\ldots,\max\{n_1,n_2\}\}|k\le z\le n_2\}=\{k,\ldots n_2\}.$$

Furthermore, $\mu_{\max\{\tilde{\chi}_1,\tilde{\chi}_2\}}(k)=\alpha'=L_{\tilde{\chi}_2}(k)$, for $n_1+1\le k\le n_2$.

Since we assume $\max\{n_1,n_2\}=n_2$, we get

$$A^1=\{z\in\{1,2,\ldots,\max\{n_1,n_2\}\}|n_2\le z\le n_2\}=\{n_2\}.$$

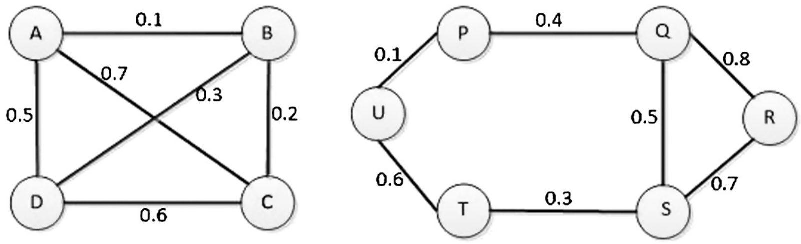

Fig. 5. Fuzzy graphs $\tilde{G}_1$ and $\tilde{G}_2$ (left and right).

Thus,

$$\begin{aligned}\max\{\tilde{\chi}_1,\tilde{\chi}_2\} = \{&(1,\min\{L_{\tilde{\chi}_1}(1),L_{\tilde{\chi}_2}(1)\}),(2,\min\{L_{\tilde{\chi}_1}(2),L_{\tilde{\chi}_2}(2)\}),\ldots,\\&(\min\{n_1,n_2\},\min\{L_{\tilde{\chi}_1}(\min\{n_1,n_2\}),L_{\tilde{\chi}_2}(\min\{n_1,n_2\})\}),\\&\ldots,(\max\{n_1,n_2\},1)\}\subseteq\tilde{\chi},\end{aligned}$$

and the proof is complete. □

A corollary can be obtained directly from Theorem 1 and Theorem 3 as follows:

Corollary 1. *If* $V_1\cap V_2\neq\emptyset$ *and* $\tilde{G}_1\subseteq\tilde{G}_2$, *then* $\max\{\tilde{\chi}_1,\tilde{\chi}_2\}=\tilde{\chi}_2$.

According to Theorem 1, Theorem 2, and Theorem 3, we have remarks as follows.

Remark 2. If $\tilde{G}_1$ and $\tilde{G}_2$ degenerate into underlying crisp graphs G_1^* and G_2^*, then union $\tilde{G}=\tilde{G}_1\cup\tilde{G}_2$ degenerates into underlying crisp graph $G^*=G_1^*\cup G_2^*$. According to (2) and the value $Q_\alpha(\tilde{\chi})\geq\max\{Q_\alpha(\tilde{\chi}_1)Q_\alpha(\tilde{\chi}_2)\}$ for a decision level $\alpha=0$, we obtain a comparison of crisp numbers $\chi(G^*)=\max\{\chi(G_1^*),\chi(G_2^*)\}$. Hence, chromatic number of union of crisp graphs is a special case of fuzzy chromatic number of union of fuzzy graphs.

Remark 3. Based on Theorem 3, we get

$$\begin{aligned}&A^{\min\{L_{\tilde{\chi}_1}(1),L_{\tilde{\chi}_2}(1)\}}=\{1,2,3,\ldots,(n_1+n_2)\}=\tilde{\chi}^{\min\{L_{\tilde{\chi}_1}(1),L_{\tilde{\chi}_2}(1)\}},\\&A^{\min\{L_{\tilde{\chi}_1}(2),L_{\tilde{\chi}_2}(2)\})\}}=\{2,3,\ldots,(n_1+n_2)\}=\tilde{\chi}^{\min\{L_{\tilde{\chi}_1}(2),L_{\tilde{\chi}_2}(2)\}},\ldots,\\&A^{\min\{L_{\tilde{\chi}_1}(\min\{n_1,n_2\}),L_{\tilde{\chi}_2}(\min\{n_1,n_2\})\}}\\&\quad=\{\min\{n_1,n_2\},\min\{n_1,n_2\}+1,\ldots,(n_1+n_2)\}\\&\quad=\tilde{\chi}^{\min\{L_{\tilde{\chi}_1}(\min\{n_1,n_2\}),L_{\tilde{\chi}_2}(\min\{n_1,n_2\})\}},\ldots,\\&A^1=\{\max\{n_1,n_2\},\max\{n_1,n_2\}+1,\ldots,(n_1+n_2)\}=\tilde{\chi}^1.\end{aligned}$$

Hence, $\max\{\max\{\tilde{\chi}_1,\tilde{\chi}_2\},\tilde{\chi}\}=\tilde{\chi}$. According to [25], $\tilde{B}\succeq\tilde{A}$ if and only if $\max\{\tilde{A},\tilde{B}\}=\tilde{B}$ for any discrete fuzzy numbers $\tilde{A}$ and $\tilde{B}$.

Thus, $\tilde{\chi}\succeq\max\{\tilde{\chi}_1,\tilde{\chi}_2\}$.

Example 2. We give an illustration of Theorem 1. Let us consider a union of two fuzzy graphs $\tilde{G}_1\cup\tilde{G}_2$ in Fig. 5.

Fuzzy chromatic numbers of fuzzy graphs $\tilde{G}_1$ and $\tilde{G}_2$ are:

$\tilde{\chi}_1=\tilde{\chi}(\tilde{G}_1)=\{(1,0.3),(2,0.5),(3,0.9),(4,1)\}$ and
$\tilde{\chi}_2=\tilde{\chi}(\tilde{G}_2)=\{(1,0.2),(2,0.5),(3,1),(4,1),(5,1),(6,1)\}$, respectively.

We can derive fuzzy chromatic number of $\tilde{\chi}(\tilde{G})$ in a simple way by using Theorem 1 as follows:

$$\tilde{\chi}(\tilde{G})=\{(1,0.2),(2,0.5),(3,0.9),(4,1),(5,1),(6,1),(7,1),(8,1),(9,1),(10,1)\}.$$

Furthermore, we can compare $\tilde{\chi}$, $\tilde{\chi}_1$, and $\tilde{\chi}_2$ in a crisp mode by using $Q_\alpha(\tilde{\chi}_i)$ for $i = 1, 2$, and $Q_\alpha(\tilde{\chi})$ for any decision level $0 \le \alpha \le 1$ as follows:

$$\begin{aligned}
Q_\alpha(\tilde{\chi}_1) &= 1(0.3-\alpha) + 2(0.5-0.3) + 3(0.9-0.5) + 4(1-0.9) = 2.3-\alpha,\\
Q_\alpha(\tilde{\chi}_2) &= 1(0.2-\alpha) + 2(0.5-0.2) + 3(1-0.5) + 4(1-1) + 5(0) + 6(0)\\
&= 2.3-\alpha,\\
Q_\alpha(\tilde{\chi}) &= 1(0.2-\alpha) + 2(0.5-0.2) + 3(0.9-0.5) + 4(1-0.9) + 5(0) +\\
&\quad \ldots + 10(0) = 2.4-\alpha.
\end{aligned}$$

Thus, $Q_\alpha(\tilde{\chi}) > \max\{Q_\alpha(\tilde{\chi}_1), Q_\alpha(\tilde{\chi}_2)\}$ for decision level $0 \le \alpha \le 1$.

Another way to compare $\tilde{\chi}$, $\tilde{\chi}_1$, and $\tilde{\chi}_2$ in a fuzzy mode is by using maximum between discrete fuzzy numbers as follows:

$$\begin{aligned}
A^{0.2} &= \{z \in \{1, 2, \ldots, 6\} | 1 \le z \le 6\} = \{1, 2, 3, 4, 5, 6\},\\
A^{0.3} &= \{z \in \{1, 2, \ldots, 6\} | 2 \le z \le 6\} = \{2, 3, 4, 5, 6\},\\
A^{0.5} &= \{z \in \{1, 2, \ldots, 6\} | 2 \le z \le 6\} = \{2, 3, 4, 5, 6\},\\
A^{0.9} &= \{z \in \{1, 2, \ldots, 6\} | 3 \le z \le 6\} = \{3, 4, 5, 6\}, \text{ and}\\
A^{1} &= \{z \in \{1, 2, \ldots, 6\} | 3 \le z \le 6\} = \{4, 5, 6\}.
\end{aligned}$$

Thus, $\max\{\tilde{\chi}_1, \tilde{\chi}_2\} = \{(1, 0.2), (2, 0.5), (3, 0.9), (4, 1), (5, 1), (6, 1)\} \subseteq \tilde{\chi}$.

Further, we compare $\max\{\tilde{\chi}_1, \tilde{\chi}_2\}$ and $\tilde{\chi}$. It is clear that

$$\begin{aligned}
A^{0.2} &= \{z \in \{1, 2, \ldots, 10\} | 1 \le z \le 10\} = \tilde{\chi}^{0.2},\\
A^{0.5} &= \{z \in \{1, 2, \ldots, 10\} | 2 \le z \le 10\} = \tilde{\chi}^{0.5},\\
A^{0.9} &= \{z \in \{1, 2, \ldots, 10\} | 3 \le z \le 10\} = \tilde{\chi}^{0.9},\\
A^{1} &= \{z \in \{1, 2, \ldots, 10\} | 4 \le z \le 10\} = \tilde{\chi}^{1}.
\end{aligned}$$

Hence, $\max\{\max\{\tilde{\chi}_1, \tilde{\chi}_2\}, \tilde{\chi}\} = \tilde{\chi} \Leftrightarrow \tilde{\chi} \succeq \max\{\tilde{\chi}_1, \tilde{\chi}_2\}$.

Also, we investigate fuzzy chromatic number of a fuzzy subgraph of $\tilde{G}(V, \tilde{E})$. We give the properties in Lemma 1 and Theorem 4.

Lemma 1. *Let $\tilde{G}_1(V_1, \tilde{E}_1)$ and $\tilde{G}_2(V_2, \tilde{E}_2)$ be two fuzzy graphs with $|V_1| = n_1$ and $|V_2| = n_2$. Fuzzy chromatic numbers of $\tilde{G}_1$ and $\tilde{G}_2$ are $\tilde{\chi}_1$ and $\tilde{\chi}_2$, respectively. If $\tilde{G}_1 \subseteq \tilde{G}_2$, then $L_{\tilde{\chi}_1}(k) \ge L_{\tilde{\chi}_2}(k)$ for $1 \le k \le n_1$.*

Proof. It follows from definition of fuzzy subgraphs that $\mu_{\tilde{E}_1}(uv) \le \mu_{\tilde{E}_2}(uv)$ for $u, v \in V_1$. Let $1 \le k \le n_1$. There exists $\delta' \in \mu_{\tilde{E}_1}(V_1 \times V_1)$ such that we can construct a partition $\{S_1^{\delta'}, S_2^{\delta'}, \ldots, S_k^{\delta'}\}$ which gives $\chi^{\delta'}(\tilde{G}_1) = k$. There also exists $\delta'' \in \mu_{\tilde{E}_2}(V_2 \times V_2)$ where $\delta'' \ge \delta'$, such that we can construct a partition $\{S_1^{\delta''}, S_2^{\delta''}, \ldots, S_k^{\delta''}\}$ which gives $\chi^{\delta''}(\tilde{G}_2) = k$. Thus, $L_{\tilde{\chi}_1}(k) = 1 - \delta' \ge 1 - \delta'' = L_{\tilde{\chi}_2}(k)$. □

Theorem 4. *Let $\tilde{G}_1(V_1, \tilde{E}_1)$ and $\tilde{G}_2(V_2, \tilde{E}_2)$ be two fuzzy graphs as described in Lemma 1 where $|V_1| = n_1$ and $|V_2| = n_2$, respectively. If $\tilde{G}_1 \subseteq \tilde{G}_2$, then $Q_\alpha(\tilde{\chi}_1) \le Q_\alpha(\tilde{\chi}_2)$ for all decision levels $\alpha \in [0, 1]$.*

Proof. Let $0 \le \alpha \le 1$. Since $\tilde{G}_1 \subseteq \tilde{G}_2$, it is obvious that $n_1 \le n_2$. Let us consider $Q_\alpha(\tilde{\chi}_1)$ and $Q_\alpha(\tilde{\chi}_2)$ as presented in Equations (4) and (5). Based on Lemma 1, we have $L_{\tilde{\chi}_1}(k) \ge L_{\tilde{\chi}_2}(k)$ for $1 \le k \le n_1$. According to Equations (4) and (5), we obtain $Q_\alpha(\tilde{\chi}_1) \le Q_\alpha(\tilde{\chi}_2)$ and the theorem is proved. □

We discuss further in the forthcoming section, an application of fuzzy chromatic number of union of fuzzy graphs on an integrated traffic light system in determining the number of phases needed as regards different traffic intensities. We model two traffic lights on two consecutive intersections into one integrated system which is needed to reduce cycle time of traffic light between both intersections.

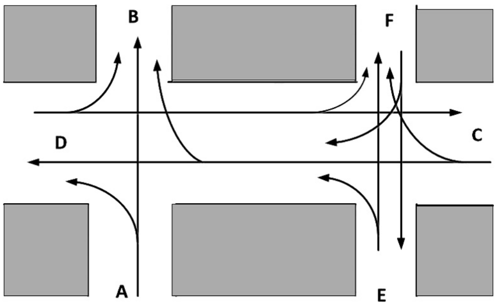

Fig. 6. A traffic light system on two intersections.

Table 2
Data of traffic flows (vehicles/hour).

	CF	FD	EF	FE	ED	CD	DC	DF	AB	AD	DB	CB
ν	316	474	437	472	217	248	467	445	476	255	253	261
Degree(l)					0.07							
Degree(m)	0.84				0.17	0.48			0.55	0.53	0.61	
Degree(h)		0.94	0.59	0.92			0.87	0.67	0.96			

4. An application

Some terminologies in traffic light systems used in this paper are as follows [1]:

1. A phase is a part of a signal cycle with a green light allocated to a specific combination of traffic movements.
2. An approach is the area of an intersection arm for vehicles to queue before being discharged across the stop line.
3. A traffic flow is the number of traffic elements passing a point on a road per unit of time (vehicles/hour or passenger car units/hour).
4. Conflict is traffic movements arriving from intersecting approaches.

Let us consider a four-way intersection (crossroads) on two intersections as illustrated in Fig. 6. There are four approaches on the first (left side) intersection, i.e., A on the south, B on the north, C on the east, and D on the west. Also, there are four approaches on the second (right side) intersection, i.e., E, F, C, and D. Therefore, there are traffic movements in different directions, i.e., CD, DC, DF, CF, FD, EF, FE, ED, CB, AB, AD, and DB. We represent the two signalized intersections through union of fuzzy graphs.

A traffic movement that goes from one approach to another is represented as a vertex. Two vertices which represent conflicting traffic movements should be connected with an edge. A membership degree assigned to an edge indicates a degree of conflict, that is a possibility for accidents to occur between vehicles from both movements. In the first step, we change data of traffic flows into three levels of fuzzy sets i.e., low, mid, or high flows. Let us take a look at data of traffic flows in Table 2.

Notation l, m, and h mean low, mid, and high. Symbol ν represents the number of vehicles on a traffic movement. We present: fuzzy set of low traffic flow by using a trapezoidal membership function at interval $[0, 225]$, fuzzy set of mid traffic flow by using a triangular membership function at interval $[200, 400]$, and high traffic flow by a trapezoidal membership function at interval $[375, 600]$. Fuzzification of traffic flow data in Table 2 is illustrated on Fig. 7.

In the second step, we determine degrees of membership of all edges (degrees of conflict) by using a rule as follows:

a) If traffic movements XY and UV are in conflict, then there is an edge $XY - UV$. Further, we choose a maximum value between the number of vehicles on XY and UV to determine the flow on the edge $XY - UV$ and calculate its membership degree.
b) If movements XY and UV are not in conflict, then there is no $XY - UV$ edge. This means that the membership degree of the edge $XY - UV$ is zero.

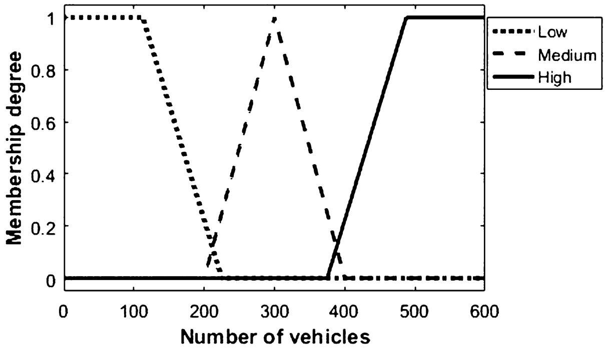

Fig. 7. Membership functions of low, mid, and high traffic flows.

Table 3
Fuzzy edge set $\tilde{E}_1$.

Movements	CF	FD	EF	FE	ED	CD	DC	DF
CF	–	0.94	0.59	0.92	–	–	0.87	–
FD	0.94	–	0.94	–	–	0.94	0.94	–
EF	0.59	0.94	–	–	–	0.59	0.87	0.67
FE	0.92	–	–	–	–	0.92	0.92	–
ED	–	–	–	–	–	0.48	–	–
CD	–	0.94	0.59	0.92	0.48	–	–	–
DC	0.87	0.94	0.87	0.92	–	–	–	–
DF	–	–	0.67	–	–	–	–	–

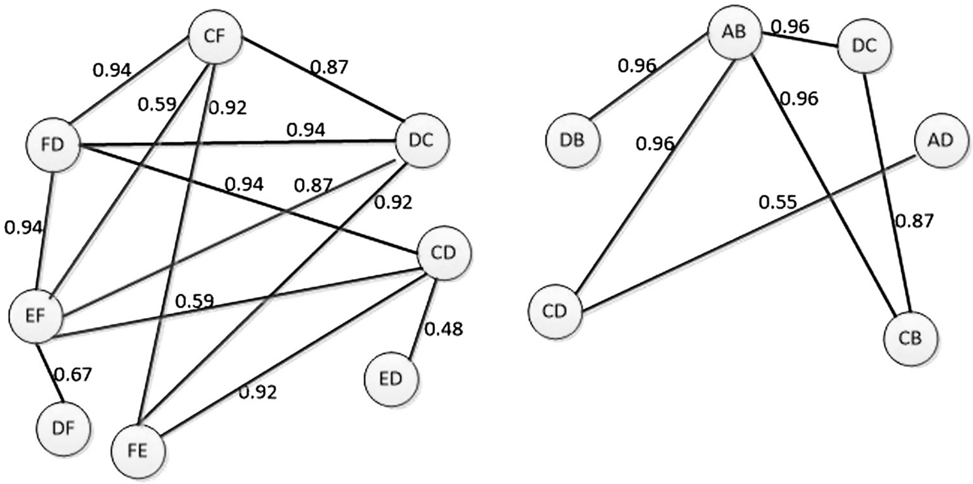

Fig. 8. Fuzzy graph models for traffic light system in Fig. 6.

By using the given rule, traffic flows on the first and second intersection in Fig. 6 can be modelled as two fuzzy graphs $\tilde{G}_1(V_1, \tilde{E}_1)$ and $\tilde{G}_2(V_2, \tilde{E}_2)$, respectively, where the vertex sets are $V_1 = \{CD, DC, DF, CF, FD, EF, FE, ED\}$ and $V_2 = \{CD, CB, AB, AD, DB, DC\}$. Membership degrees of edges in fuzzy edge set $\tilde{E}_1$ is presented in Table 3. Whereas, fuzzy edge set $\tilde{E}_2$ can be obtained similarly. Fuzzy graphs $\tilde{G}_1(V_1, \tilde{E}_1)$ and $\tilde{G}_2(V_2, \tilde{E}_2)$ are shown in Fig. 8.

By using fuzzy chromatic algorithm, we get fuzzy chromatic numbers of fuzzy graphs $\tilde{G}_1$ and $\tilde{G}_2$ as follows:

$$\tilde{\chi}(\tilde{G}_1) = \{(1, 0.06), (2, 0.13), (3, 0.41), (4, 1), (5, 1), (6, 1), (7, 1), (8, 1)\} \text{ and}$$

$$\tilde{\chi}(\tilde{G}_2) = \{(1, 0.04), (2, 0.13), (3, 1), (4, 1), (5, 1), (6, 1)\}.$$

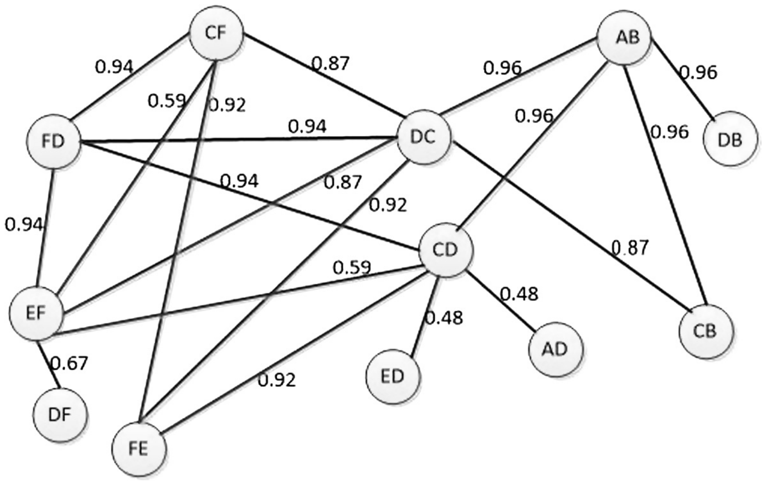

Fig. 9. The model of union of fuzzy graphs in Fig. 8.

Table 4
Traffic light arrangements by using k-phases.

δ	k	$L_{\tilde{\chi}}(k)$	Arrangements (partitions of vertex set V)
0	4	1	$\{DC, CD, DB, DF\}, \{EF, FE, AD, CB\}, \{CF, ED, AB\}, \{FD\}$
0.59	3	0.41	$\{CF, EF, ED, AB\}, \{CD, DC, DB, AD\}, \{FE, DF, FD, CB\}$
0.87	2	0.13	$\{CF, EF, DC, CD, ED, AD, CB, DB\}, \{FD, FE, DF, AB\}$
0.96	1	0.04	$V = V_1 \cup V_2$

Furthermore, two intersections in Fig. 6 can be modelled into an integrated traffic light system by using union of fuzzy graphs $\tilde{G} = \tilde{G}_1 \cup \tilde{G}_2$ as illustrated in Fig. 9. According to Theorem 1, we get fuzzy chromatic number of union $\tilde{G} = \tilde{G}_1 \cup \tilde{G}_2$ in Fig. 9 as follows:

$$\tilde{\chi}(\tilde{G}) = \{(1, 0.04), (2, 0.13), (3, 0.41), (4, 1), (5, 1), (6, 1), \ldots, (12, 1)\}. \tag{6}$$

Fuzzy chromatic number of union of fuzzy graphs $\tilde{\chi}(\tilde{G}) = \{(k, L_{\tilde{\chi}}(k))\}$ can be interpreted as follows: the number k represents the number of phases needed on the integrated system and degree of membership $L_{\tilde{\chi}}(k)$ represents possibility that there are no accidents (degree of safety) when we use k phases.

Based on fuzzy chromatic number in (6) and δ-chromatic numbers k, traffic flows in Fig. 6 could be arranged in particular patterns as shown in Table 4. For example, when we make use of $k = 2$ phases, then 0.13 degree of safety is reached. Traffic flows allowed to move simultaneously (get the green light) on the first phase are CF, EF, DC, CD, ED, AD, CB, and DB. Whereas, those allowed on the second phase are FD, FE, DF, and AB.

The objective is to get an optimal arrangement. The optimality is reached when the degree of safety is high while the number of phases is small.

5. Conclusions

A modified fuzzy chromatic algorithm to determine fuzzy chromatic number of fuzzy graphs has been developed in this paper. The algorithm applied on Matlab shows a constant running time over 150 trials. Also, the algorithm has been presented with its complexity. Further, we have generalized chromatic number of union of crisp graphs into fuzzy chromatic number of union of fuzzy graphs. Moreover, we have characterized a connection between fuzzy chromatic number of union of fuzzy graphs $\tilde{\chi}(\tilde{G}) = \tilde{\chi}(\tilde{G}_1 \cup \tilde{G}_2)$ and maximum of fuzzy chromatic numbers $\{\tilde{\chi}_1(\tilde{G}_1), \tilde{\chi}_2(\tilde{G}_2)\}$ by using two approaches. The first approach is comparing fuzzy chromatic numbers through defuzzification on all decision levels $\alpha \in [0, 1]$, while the second approach is comparing fuzzy chromatic numbers through their α-cuts.

Finally, we have proposed an application of fuzzy chromatic number of union of fuzzy graphs to determine the number of phases of an integrated traffic light system (consisting of two signalized intersections). We might use different phases with different degrees of safety depending on traffic intensities at the intersections.

In our upcoming research, we will examine an algorithm to model an integrated traffic light system while considering green light durations. Also, we will verify some properties of fuzzy chromatic number of union of fuzzy graphs based on α-cut graph coloring.

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